This worksheet is a self-paced and ungraded way for you to check your understanding of linear algebra. You do not have to turn it in; the worksheet is meant to support you and ease you into the course content. The material on this worksheet is covered in prerequisite courses. If it’s not familiar to you, consider reviewing the appropriate material from the prerequisites.

Please attempt each problem before looking at its solution. This will help you understand the content better than just reading the solutions.

1. Subspaces and Dimensions
   Consider the set $S$ of points $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that
   \[
   x_1 + 2x_2 + 3x_3 = 0, \quad 3x_1 + 2x_2 + x_3 = 0.
   \]
   (1)
   (a) Show that $S$ is a subspace of $\mathbb{R}^3$.
   (b) Find a $2 \times 3$ matrix $A$ for which $S$ is exactly the null space of $A$.
   (c) Determine the dimension of $S$ and find a basis for it.

2. Orthogonality
   Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ be two linearly independent unit-norm vectors; that is, $\|\vec{x}\|_2 = \|\vec{y}\|_2 = 1$.
   (a) Show that the vectors $\vec{u} = \vec{x} - \vec{y}$ and $\vec{v} = \vec{x} + \vec{y}$ are orthogonal.
   (b) Find an orthonormal basis for $\text{span}(\vec{x}, \vec{y})$, the subspace spanned by $\vec{x}$ and $\vec{y}$.

3. Eigenvalues
   Let $A \in \mathbb{R}^{n \times n}$ have the eigendecomposition $P\Lambda P^{-1}$ where $\Lambda \in \mathbb{R}^{n \times n}$ is a diagonal matrix with entries consisting of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $P \in \mathbb{R}^{n \times n}$ is an invertible matrix. Note that this is equivalent to stating that $A$ is diagonalizable via the transformation,
   \[
P^{-1}AP = \Lambda.
   \]
   (2)
   (a) Show that $A^m = PA^mP^{-1}$, for integer $m \geq 1$.
   (b) Show that determinant of $A$ is the product of its eigenvalues, i.e.
   \[
   \det(A) = \prod_{i=1}^{n} \lambda_i.
   \]
   (HINT: We have the identity $\det(XY) = \det(X)\det(Y)$.)