Discussion 1

1. Eigenvalues

Let $A \in \mathbb{R}^{n \times n}$ have the eigendecomposition $P \Lambda P^{-1}$ where $\Lambda \in \mathbb{R}^{n \times n}$ is a diagonal matrix with entries consisting of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $P \in \mathbb{R}^{n \times n}$ is an invertible matrix. Note that this is equivalent to stating that A is diagonalizable via the transformation,

$$P^{-1} A P = \Lambda. \tag{1}$$

(a) Show that $A^m = P \Lambda^m P^{-1}$, for integer $m \geq 1$.

(b) Show that determinant of A is the product of its eigenvalues, i.e.

$$\det(A) = \prod_{i=1}^{n} \lambda_i. \tag{2}$$

HINT: We have the identity $\det(XY) = \det(X) \det(Y)$.

2. Invertibility of $A^\top A$

In this problem, we show that if the matrix $A \in \mathbb{R}^{m \times n}$ has a full column rank, then the matrix $A^\top A$ is invertible.

(a) Show that if a vector $\vec{x}$ is in the null space of A then $\vec{x}$ is in the null space of $A^\top A$.

(b) Conversely, show that if $\vec{x}$ is in the null space of $A^\top A$ then $\vec{x}$ is in the null space of A.

(c) Given that matrix A has a full column rank, what can you say about its null space? What does this imply about the null space and invertibility of the matrix $A^\top A$?