1. Matrix Decomposition and Inverses
   Let $A \in \mathbb{R}^{3 \times 2}$ have full column rank and let its singular value decomposition (SVD) be $U \Sigma V^T$, where all non-diagonal entries of $\Sigma \in \mathbb{R}^{3 \times 2}$ are 0. Here, $U \in \mathbb{R}^{3 \times 3}$ and $V \in \mathbb{R}^{2 \times 2}$ are matrices such that the columns of each form an orthonormal set.

   (a) Show that $A^T A$ is invertible.

   (b) Let $B = (A^T A)^{-1} A^T$. Write $B$ in terms of $U, V$, and the diagonal entries of $\Sigma$. Show that $BA = I$.

   (c) Show that $A(A^T A)^{-1} A^T$ is a projection matrix that projects $\mathbb{R}^3$ onto the column space of $A$.

   (d) Now suppose $A \in \mathbb{R}^{2 \times 2}$ instead and $A$ is full rank. Express $A^{-1}$ in terms of $U, V$, and the diagonal entries of $\Sigma$.

2. Least Squares and Gram-Schmidt
   Consider the least squares problem
   \[
   \hat{x}^* = \arg\min_{\hat{x} \in \mathbb{R}^n} \| A\hat{x} - \hat{b} \|_2^2
   \]
   where $A \in \mathbb{R}^{m \times n}$, $\hat{b} \in \mathbb{R}^m$ and assume $A$ is full column rank. One way to solve this least-squares problems is to use Gram-Schmidt Orthonormalization (GSO). Using GSO, the matrix $A$ can be written as,
   \[
   A = QR = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \end{bmatrix}
   \]
where $Q$ is an orthonormal matrix and $R$ is an upper-triangular matrix. The columns of $Q_1$ form an orthonormal basis for the range space $\mathcal{R}(A)$ and columns of $Q_2$ form an orthonormal basis for the range space $\mathcal{R}(A)^\perp$. Moreover, $R_1$ is upper triangular and invertible.

(a) Show that the squared norm of the residual is given by

$$
\|\vec{r}\|_2^2 := \|\vec{b} - A\vec{x}\|_2^2 = \|Q_1^\top \vec{b} - R_1 \vec{x}\|_2^2 + \|Q_2^\top \vec{b}\|_2^2. \quad (3)
$$

(b) Find $\vec{x}^*$ such that the squared norm of the residual in Equation (3) is minimized. Your expression for $\vec{x}^*$ should only use some or all of the following terms: $Q_1$, $Q_2$, $R_1$, $\vec{b}$.

(c) Check if the expression for $\vec{x}^*$ obtained in the previous part is equivalent to the one obtained by the formula, $\vec{x}^* = (A^\top A)^{-1} A^\top \vec{b}$.