1. Eigenvalues

Let $A \in \mathbb{R}^{n \times n}$ have the eigendecomposition $P \Lambda P^{-1}$ where $\Lambda \in \mathbb{R}^{n \times n}$ is a diagonal matrix with entries consisting of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $P \in \mathbb{R}^{n \times n}$ is an invertible matrix. Note that this is equivalent to stating that $A$ is diagonalizable via the transformation,

$$P^{-1}AP = \Lambda.$$  \hfill (1)

(a) Show that $A^m = P \Lambda^m P^{-1}$, for integer $m \geq 1$.

**Solution:**

$$A^m = (P \Lambda P^{-1})(P \Lambda P^{-1}) \ldots (P \Lambda P^{-1}) \quad m \text{ times} \hfill (2)$$

$$= P \Lambda (P^{-1} P) \Lambda (P^{-1} P) \ldots \Lambda (P^{-1} P) \Lambda P^{-1} \hfill (3)$$

$$= P \Lambda^m P^{-1}.$$ \hfill (4)

The last equality follows from the repeated application of the identity $P^{-1} P = I$.

(b) Show that determinant of $A$ is the product of its eigenvalues, i.e.

$$\det(A) = \prod_{i=1}^{n} \lambda_i.$$ \hfill (5)

**HINT:** We have the identity $\det(XY) = \det(X) \det(Y)$.

**Solution:** Write down eigendecomposition of $A$ and use properties of determinant given in the hint.

$$\det(A) = \det(P \Lambda P^{-1})$$ \hfill (6)

$$= \det(P) \det(\Lambda) \det(P^{-1})$$ \hfill (7)

$$= \det(P P^{-1}) \det(\Lambda)$$ \hfill (8)

$$= \det(\Lambda)$$ \hfill (9)

$$= \prod_{i=1}^{n} \lambda_i$$ \hfill (10)
2. Invertibility of $A^\top A$

In this problem, we show that if the matrix $A \in \mathbb{R}^{m \times n}$ has a full column rank, then the matrix $A^\top A$ is invertible.

(a) Show that if a vector $\vec{x}$ is in the null space of $A$ then $\vec{x}$ is in the null space of $A^\top A$.

Solution:

$$\vec{x} \in \mathcal{N}(A) \iff A\vec{x} = \vec{0}$$  \hspace{1cm} (11)
$$\implies A^\top A \vec{x} = \vec{0}$$  \hspace{1cm} (12)
$$\iff \vec{x} \in \mathcal{N}(A^\top A)$$  \hspace{1cm} (13)

Where line 12 follows by multiplying both sides of $A\vec{x} = \vec{0}$ by $A^\top$

(b) Conversely, show that if $\vec{x}$ is in the null space of $A^\top A$ then $\vec{x}$ is in the null space of $A$.

Solution:

$$\vec{x} \in \mathcal{N}(A^\top A) \iff A^\top A \vec{x} = \vec{0}$$  \hspace{1cm} (14)
$$\implies \vec{x}^\top A^\top A \vec{x} = 0$$  \hspace{1cm} (15)
$$\implies (A\vec{x})^\top A\vec{x} = 0$$  \hspace{1cm} (16)
$$\implies \|A\vec{x}\|_2^2 = 0$$  \hspace{1cm} (17)
$$\implies A\vec{x} = \vec{0}$$  \hspace{1cm} (18)
$$\implies \vec{x} \in \mathcal{N}(A)$$  \hspace{1cm} (19)

Where line 15 follows by multiplying both sides of $A^\top A \vec{x} = \vec{0}$ by $\vec{x}^\top$ and line 18 follows from the properties of norms.

(c) Given that matrix $A$ has a full column rank, what can you say about its null space? What does this imply about the null space and invertibility of the matrix $A^\top A$?

Solution: $\mathcal{N}(A) = \{\vec{0}\}$. From the previous parts, we have shown that $\mathcal{N}(A) = \mathcal{N}(A^\top A)$ then $\mathcal{N}(A^\top A) = \{\vec{0}\}$ and thus $A^\top A$ is invertible.