Discussion 1

1. Eigenvalues

Let $A \in \mathbb{R}^{n\times n}$ have the eigendecomposition $P\Lambda P^{-1}$ where $\Lambda \in \mathbb{R}^{n\times n}$ is a diagonal matrix with entries consisting of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $P \in \mathbb{R}^{n\times n}$ is an invertible matrix. Note that this is equivalent to stating that A is diagonalizable via the transformation,

$$P^{-1}AP = \Lambda. \tag{1}$$

(a) Show that $A^m = P\Lambda^m P^{-1}$, for integer $m \geq 1$.

Solution:

$$\begin{aligned} A^m &= (P\Lambda P^{-1})(P\Lambda P^{-1})\ldots(P\Lambda P^{-1}) \quad m \text{ times} && (2)\\ &= P\Lambda(P^{-1}P)\Lambda(P^{-1}P)\ldots\Lambda(P^{-1}P)\Lambda P^{-1} && (3)\\ &= P\Lambda^m P^{-1}. && (4)\end{aligned}$$

The last equality follows from the repeated application of the identity $P^{-1}P = I$.

(b) Show that determinant of A is the product of its eigenvalues, i.e.

$$\det(A) = \prod_{i=1}^{n} \lambda_i. \tag{5}$$

HINT: We have the identity $\det(XY) = \det(X)\det(Y)$.

Solution: Write down eigendecomposition of A and use properties of determinant given in the hint.

$$\begin{aligned} \det(A) &= \det\left(P\Lambda P^{-1}\right) && (6)\\ &= \det(P)\det(\Lambda)\det\left(P^{-1}\right) && (7)\\ &= \det\left(PP^{-1}\right)\det(\Lambda) && (8)\\ &= \det(\Lambda) && (9)\\ &= \prod_{i=1}^{n}\lambda_i && (10)\end{aligned}$$

2. Invertibility of $A^\top A$

In this problem, we show that if the matrix $A \in \mathbb{R}^{m\times n}$ has a full column rank, then the matrix $A^\top A$ is invertible.

(a) Show that if a vector $\vec{x}$ is in the null space of A then $\vec{x}$ is in the null space of $A^\top A$.

Solution:

$$\vec{x} \in \mathcal{N}(A) \iff A\vec{x} = \vec{0} \tag{11}$$
$$\implies A^\top A\vec{x} = \vec{0} \tag{12}$$
$$\iff \vec{x} \in \mathcal{N}\left(A^\top A\right) \tag{13}$$

Where line 12 follows by multiplying both sides of $A\vec{x} = 0$ by $A^\top$

(b) Conversely, show that if $\vec{x}$ is in the null space of $A^\top A$ then $\vec{x}$ is in the null space of A.

Solution:

$$\vec{x} \in \mathcal{N}\left(A^\top A\right) \iff A^\top A\vec{x} = \vec{0} \tag{14}$$
$$\implies \vec{x}^\top A^\top A\vec{x} = 0 \tag{15}$$
$$\implies (A\vec{x})^\top A\vec{x} = 0 \tag{16}$$
$$\implies \|A\vec{x}\|_2^2 = 0 \tag{17}$$
$$\implies A\vec{x} = \vec{0} \tag{18}$$
$$\implies \vec{x} \in \mathcal{N}(A) \tag{19}$$

Where line 15 follows by multiplying both sides of $A^\top A\vec{x} = \vec{0}$ by $\vec{x}^\top$ and line 18 follows from the properties of norms.

(c) Given that matrix A has a full column rank, what can you say about its null space? What does this imply about the null space and invertibility of the matrix $A^\top A$?

Solution: $\mathcal{N}(A) = \{\vec{0}\}$. From the previous parts, we have shown that $\mathcal{N}(A) = \mathcal{N}\left(A^\top A\right)$ then $\mathcal{N}\left(A^\top A\right) = \{\vec{0}\}$ and thus $A^\top A$ is invertible.