Discussion 2

1. **Proof of the Fundamental Theorem of Linear Algebra**

In this question, we will prove the fundamental theorem of linear algebra. For any $A \in \mathbb{R}^{m \times n}$, let $\mathcal{N}(A)$, $\mathcal{R}(A)$, and $\operatorname{rank}(A)$ denote the null space, range and rank of A respectively. For any subspace, $\mathcal{S}$ with dimension, $\dim(\mathcal{S})$, let $\mathcal{S}^\perp$ denote its the subspace orthogonal to $\mathcal{S}$. The fundamental theorem of linear algebra states that,

$$\mathcal{N}(A) \oplus \mathcal{R}\left(A^\top\right) = \mathbb{R}^n. \tag{1}$$

The proof technique we employ will first show that,

$$\mathcal{N}(A) = \mathcal{R}\left(A^\top\right)^\perp. \tag{2}$$

Then we will prove that we can find orthonormal vectors $\vec{e}_1, \vec{e}_2, \ldots \vec{e}_n$ such that $\mathcal{N}(A) = \operatorname{span}(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_\ell)$ and $\mathcal{R}\left(A^\top\right) = \operatorname{span}(\vec{e}_{\ell+1}, \vec{e}_{\ell+2}, \ldots, \vec{e}_n)$. As a corollary we get the rank-nullity theorem:

$$\dim(\mathcal{N}(A)) + \operatorname{rank}(A) = n. \tag{3}$$

(a) First, show that $\mathcal{N}(A) \subseteq \mathcal{R}\left(A^\top\right)^\perp$.
HINT: Consider $\vec{u}$ in $\mathcal{N}(A), \vec{v} \in \mathcal{R}\left(A^\top\right)$ and show that $\vec{u}^\top \vec{v} = 0$.

(b) Now show that: $\mathcal{R}\left(A^\top\right)^\perp \subseteq \mathcal{N}(A)$.
HINT: Show that any vector $\vec{v}$ that is orthogonal to all vectors in the range of $A^\top$ satisfies $A\vec{v} = 0$. To do this, consider $\vec{v} \in R(A^\top)^\perp$ and what it implies for $\vec{v}^\top A^\top$.

(c) Let $\dim(\mathcal{N}(A)) = \ell$ and let $\vec{e}_1, \ldots, \vec{e}_\ell$ be an orthonormal basis for $\mathcal{N}(A)$. Consider an extension of the basis to an orthonormal basis, $\vec{e}_1, \ldots, \vec{e}_n$ for $\mathbb{R}^n$. We will prove that $\vec{e}_{\ell+1}, \ldots, \vec{e}_n$ form a basis for $\mathcal{R}\left(A^\top\right)$ and as a consequence, the dimension of $\mathcal{R}\left(A^\top\right)$ is $n - \ell$.

i. Show that $\mathcal{R}\left(A^\top\right)$ lies in the span of $\vec{e}_{\ell+1}, \ldots, \vec{e}_n$.
HINT: Express any vector $\vec{u} \in \mathcal{R}\left(A^\top\right)$ *as* $\vec{u} = \sum_{i=1}^{n} \alpha_i \vec{e}_i$. *What are the values of* α_i*?*

ii. From part (i) we know that $\mathcal{R}\left(A^\top\right) \subseteq \operatorname{span}(\vec{e}_{\ell+1}, \ldots, \vec{e}_n)$, but we want something stronger. Show that in fact $\mathcal{R}\left(A^\top\right) = \operatorname{span}(\vec{e}_{\ell+1}, \ldots, \vec{e}_n)$.
HINT: First, prove $\dim\left(\mathcal{R}\left(A^\top\right)\right) = \dim(\operatorname{span}(\vec{e}_{\ell+1}, \ldots, \vec{e}_n)) = n - \ell$ *by contradiction. Assume* $\dim\left(\mathcal{R}\left(A^\top\right)\right) = k < n - \ell$, *show that a vector* $\vec{u} \in \operatorname{span}(\vec{e}_{\ell+1}, \ldots, \vec{e}_n)$ *and* $\vec{u} \notin \mathcal{R}\left(A^\top\right)$ *cannot exist.*
Specifically, let $\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k$ *be an orthonormal basis for* $\mathcal{R}\left(A^\top\right)$, *we can find non-zero* $\vec{u}_\perp = \vec{u} - \sum_{i=1}^{k} (\vec{f}_i^\top \vec{u}) \vec{f}_i$ *that is orthogonal to* $\mathcal{R}\left(A^\top\right)$. *Does* $\vec{u}_\perp$ *lie in* $\mathcal{N}(A)$*? Does* $\vec{u}_\perp$ *also lie in* $\operatorname{span}(\vec{e}_{\ell+1}, \ldots, \vec{e}_n)$*? Does this lead to a contradiction? Think of* $n - \ell = 3$ *and* $k = 2$ *for visualization.*
HINT: Second, you can use without proof the fact that for two subspaces, $\mathcal{S}_1$ *and* $\mathcal{S}_2$, *if* $\mathcal{S}_1 \subseteq \mathcal{S}_2$ *and* $\dim(\mathcal{S}_1) = \dim(\mathcal{S}_2)$ *then* $\mathcal{S}_1 = \mathcal{S}_2$.

(d) Using part (c) argue why $\mathcal{N}(A) \oplus \mathcal{R}\left(A^{\top}\right) = \mathbb{R}^n$ and why the rank nullity theorem holds.

2. Symmetric Matrices

(a) Show that any symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive semidefinite if and only if there exists a symmetric matrix $C \in \mathbb{R}^{n \times n}$ such that $A = C^{\top} C$.

(b) Draw the region $\left\{ \vec{x} \in \mathbb{R}^2 \middle| \vec{x}^{\top} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \vec{x} \leq 1 \right\}$.

(c) Draw the region $\left\{ \vec{x} \in \mathbb{R}^2 \middle| \vec{x}^{\top} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \vec{x} \leq 1 \right\}$.

(d) Why is the region in part (b) bounded, whereas the region in part (c) is unbounded?

