EECS 127/227AT Optimization Models in Engineering UC Berkeley Spring 2024

Discussion 3

1. SVD

Suppose we have a matrix $A \in \mathbb{R}^{m \times n}$ with rank r. It turns out that its SVD has multiple forms, all of which can be useful depending on the problem we're working on.

We define the compact SVD as follows:

$$\underbrace{A}_{m\times n} = \underbrace{U_r}_{m\times r}\underbrace{\Sigma_r}_{r\times r}\underbrace{V_r^\top}_{r\times n}.$$

Here, $\Sigma_r \in \mathbb{R}^{r\times r}$ is a diagonal matrix containing non-zero singular values of A.

$$\Sigma_r = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix},$$

with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$.

Next, $U_r \in \mathbb{R}^{m\times r}$ is given by,

$$U_r = \begin{bmatrix} \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r \end{bmatrix},$$

where u_i is a left singular vector corresponding to non-zero singular value, σ_i, for $i = 1, 2, \ldots, r$. The columns of U_r are orthonormal and together they span the columnspace of A.

Finally, $V_r^\top \in \mathbb{R}^{r\times n}$ is given by,

$$V_r^\top = \begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \\ \vdots \\ \vec{v}_r^\top \end{bmatrix},$$

where $\vec{v}_j$ is a right singular vector corresponding to non-zero singular value, σ_j for $j = 1, 2, \ldots, r$. The rows of $V_r^\top$ are orthonormal and span the rowspace of A. Equivalently the columns of V_r span the column space of $A^\top$.

The matrix A can be expressed as,

$$A = \sigma_1\vec{u}_1\vec{v}_1^\top + \sigma_2\vec{u}_2\vec{v}_2^\top + \ldots + \sigma_r\vec{u}_r\vec{v}_r^\top.$$

This is called the dyadic SVD, since it's expressed as the sum of dyads (matrices of the form $uv^\top$). Assume now that $m \geq n$.

Another type of SVD which might be more familiar is the full SVD of A which is defined as follows:

$$\underbrace{A}_{m\times n} = \underbrace{U}_{m\times m}\underbrace{\Sigma}_{m\times n}\underbrace{V^\top}_{n\times n}.$$

Here, $\Sigma \in \mathbb{R}^{m \times n}$ has non-diagonal entries as zero. The diagonal entries of Σ contain the singular values and we can write Σ in terms of Σ_r as,

$$\Sigma = \left[\begin{array}{c|c} \Sigma_r & 0_{r \times (n-r)} \\ \hline 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{array}\right]$$

Next, $U \in \mathbb{R}^{m \times m}$ is an orthogonal matrix. U can be expressed in terms of U_r as,

$$U = \underbrace{[U_r}_{m \times r} \quad \underbrace{\vec{u}_{r+1} \quad \dots \quad \vec{u}_m]}_{m \times (m-r)}$$

The columns $\vec{u}_{r+1}, \vec{u}_{r+2}, \dots, \vec{u}_n$ are left singular vectors corresponding to singular value 0, and together span the nullspace of $A^\top$.

Finally, $V^\top$ is an orthogonal matrix and can be expressed in terms of $V_r^\top$ as,

$$V^\top = \begin{bmatrix} V_r^\top \\ \vec{v}_{r+1}^\top \\ \vdots \\ \vec{v}_n^\top \end{bmatrix} \begin{array}{l} \} \quad r \times n \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \quad (n-r) \times n \end{array}$$

The rows $\vec{v}_{r+1}^\top, \vec{v}_{r+2}^\top, \dots, \vec{v}_n^\top$ when transposed are the right singular vectors corresponding to singular value 0, and together they span the nullspace of A.

(a) For this problem assume that $m > n > r$. Label each of the following as True or False:

(a) $UU^\top = I$

(b) $U^\top U = I$

(c) $V^\top V = I$

(d) $VV^\top = I$

(e) $U_r^\top U_r = I$

(f) $U_r U_r^\top = I$

(g) $V_r V_r^\top = I$

(h) $V_r^\top V_r = I$

(b) Find the compact SVD of A, given that it has the following full SVD:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(c) Find the full SVD of A, given that it has the following compact SVD:

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. SVD Part 2

Consider A to be the 4×3 matrix

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \tag{1}$$

where $\vec{a}_i$ for $i \in \{1, 2, 3\}$ form a set of *orthogonal* vectors satisfying $\|\vec{a}_1\|_2 = 3$, $\|\vec{a}_2\|_2 = 2$, $\|\vec{a}_3\|_2 = 1$.

(a) What is the SVD of A? Express it as $A = U\Sigma V^\top$, with Σ the diagonal matrix of singular values ordered in decreasing fashion, and explicitly describe U and V.

(b) What is the dimension of the null space, $\dim(\mathcal{N}(A))$?

(c) What is the rank of A, $\text{rank}(A)$? Provide an orthonormal basis for the range of A.

(d) Let I_3 denote the 3×3 identity matrix. Consider the matrix $\tilde{A} = \begin{bmatrix} A \\ I_3 \end{bmatrix} \in \mathbb{R}^{7\times 3}$. What are the singular values of $\tilde{A}$ (in terms of the singular values of A)?