1. SVD

Suppose we have a matrix $A \in \mathbb{R}^{m \times n}$ with rank $r$. It turns out that its SVD has multiple forms, all of which can be useful depending on the problem we’re working on.

We define the compact SVD as follows:

$$A_{m \times n} = U_r \Sigma_r V_r^T.$$  

Here, $\Sigma_r \in \mathbb{R}^{r \times r}$ is a diagonal matrix containing non-zero singular values of $A$.

$$\Sigma_r = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \sigma_r \end{bmatrix},$$

with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$.

Next, $U_r \in \mathbb{R}^{m \times r}$ is given by,

$$U_r = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r],$$

where $\vec{u}_i$ is a left singular vector corresponding to non-zero singular value, $\sigma_i$, for $i = 1, 2, \ldots, r$. The columns of $U_r$ are orthonormal and together they span the columnspace of $A$.

Finally, $V_r^T \in \mathbb{R}^{r \times n}$ is given by,

$$V_r^T = \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vdots \\ \vec{v}_r^T \end{bmatrix},$$

where $\vec{v}_j$ is a right singular vector corresponding to non-zero singular value, $\sigma_j$ for $j = 1, 2, \ldots, r$. The rows of $V_r^T$ are orthonormal and span the rowspace of $A$. Equivalently the columns of $V_r$ span the column space of $A^T$.

The matrix $A$ can be expressed as,

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \ldots + \sigma_r \vec{u}_r \vec{v}_r^T.$$  

This is called the dyadic SVD, since it’s expressed as the sum of dyads (matrices of the form $uv^T$). Assume now that $m \geq n$.

Another type of SVD which might be more familiar is the full SVD of $A$ which is defined as follows:

$$A_{m \times n} = U_{m \times m} \Sigma_{m \times n} V_{n \times n}^T.$$

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Here, $\Sigma \in \mathbb{R}^{m \times n}$ has non-diagonal entries as zero. The diagonal entries of $\Sigma$ contain the singular values and we can write $\Sigma$ in terms of $\Sigma_r$ as,

$$\Sigma = \begin{bmatrix} \Sigma_r & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix}$$

Next, $U \in \mathbb{R}^{m \times m}$ is an orthogonal matrix. $U$ can be expressed in terms of $U_r$ as,

$$U = \begin{bmatrix} U_r \\ \bar{\mathbf{u}}_{r+1} \\ \ldots \\ \bar{\mathbf{u}}_m \end{bmatrix}_{m \times (m - r)}$$

The columns $\bar{\mathbf{u}}_{r+1}, \bar{\mathbf{u}}_{r+2}, \ldots, \bar{\mathbf{u}}_m$ are left singular vectors corresponding to singular value 0, and together span the nullspace of $A^\top$.

Finally, $V^\top$ is an orthogonal matrix and can be expressed in terms of $V_r^\top$ as,

$$V^\top = \begin{bmatrix} V_r^\top \\ \bar{\mathbf{v}}_{r+1}^\top \\ \vdots \\ \bar{\mathbf{v}}_n^\top \end{bmatrix}_{(n - r) \times n}$$

The rows $\bar{\mathbf{v}}_{r+1}^\top, \bar{\mathbf{v}}_{r+2}^\top, \ldots, \bar{\mathbf{v}}_n^\top$ when transposed are the right singular vectors corresponding to singular value 0, and together they span the nullspace of $A$.

(a) For this problem assume that $m > n > r$. Label each of the following as True or False:

(a) $UU^\top = I$

(b) $U^\top U = I$

(c) $V^\top V = I$

(d) $VV^\top = I$

(e) $U_r^\top U_r = I$

(f) $U_r^\top U_r^\top = I$
(g) $V_r V_r^\top = I$

(h) $V_r^\top V_r = I$

(b) Find the compact SVD of $A$, given that it has the following full SVD:

$$A = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
2 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
1 & 0 \\
0 & 1
\end{bmatrix}.$$

(c) Find the full SVD of $A$, given that it has the following compact SVD:

$$A = \begin{bmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
2 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
1 & 0 \\
0 & 1
\end{bmatrix}.$$
2. SVD Part 2

Consider $A$ to be the $4 \times 3$ matrix

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$$  

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where $\vec{a}_i$ for $i \in \{1, 2, 3\}$ form a set of orthogonal vectors satisfying $\|\vec{a}_1\|_2 = 3$, $\|\vec{a}_2\|_2 = 2$, $\|\vec{a}_3\|_2 = 1$.

(a) What is the SVD of $A$? Express it as $A = U\Sigma V^\top$, with $\Sigma$ the diagonal matrix of singular values ordered in decreasing fashion, and explicitly describe $U$ and $V$.

(b) What is the dimension of the null space, $\text{dim}(\mathcal{N}(A))$?

(c) What is the rank of $A$, $\text{rank}(A)$? Provide an orthonormal basis for the range of $A$.

(d) Let $I_3$ denote the $3 \times 3$ identity matrix. Consider the matrix $\tilde{A} = \begin{bmatrix} A \\ I_3 \end{bmatrix} \in \mathbb{R}^{7 \times 3}$. What are the singular values of $\tilde{A}$ (in terms of the singular values of $A$)?