1. Convexity of Functions

**Definition.** A function \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) is convex if \( \text{dom}(f) \) is a convex set and if for all \( \vec{x}, \vec{y} \in \text{dom}(f) \) and \( \theta \in [0, 1] \), we have,
\[
f(\theta \vec{x} + (1 - \theta) \vec{y}) \leq \theta f(\vec{x}) + (1 - \theta) f(\vec{y}).
\]
(1)

The function \( f \) is strictly convex if the inequality is strict.

**Definition.** A function \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) is concave if \( \text{dom}(f) \) is a convex set and if for all \( \vec{x}, \vec{y} \in \text{dom}(f) \) and \( \theta \) with \( 0 \leq \theta \leq 1 \), we have,
\[
f(\theta \vec{x} + (1 - \theta) \vec{y}) \geq \theta f(\vec{x}) + (1 - \theta) f(\vec{y}).
\]
(2)

The function \( f \) is strictly concave if the inequality is strict.

**Property.** A function \( f \) is concave if and only if \(-f\) is convex. An affine function is both convex and concave.

**Property: Jensen’s inequality.** The inequality in Equation (1) is known as **Jensen’s Inequality.** This can be extended to convex combinations of more than one point. If \( f \) is convex, and \( \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in \text{dom}(f) \), and \( \theta_1, \theta_2, \ldots, \theta_k \geq 0 \) with \( \sum_{i=1}^{k} \theta_i = 1 \) then,
\[
f(\theta_1 \vec{x}_1 + \theta_2 \vec{x}_2 + \cdots + \theta_k \vec{x}_k) \leq \theta_1 f(\vec{x}_1) + \theta_2 f(\vec{x}_2) + \cdots + \theta_k f(\vec{x}_k).
\]
(3)

**Property: first order condition.** Suppose \( f \) is differentiable. Then \( f \) is convex if and only if \( \text{dom}(f) \) is convex and
\[
f(\vec{y}) \geq f(\vec{x}) + \nabla f(\vec{x})^\top (\vec{y} - \vec{x}),
\]
(4)

for all \( \vec{x}, \vec{y} \in \text{dom}(f) \).

**Property: Second order condition.** Suppose \( f \) is twice differentiable. Then \( f \) is convex if and only if, \( \text{dom}(f) \) is convex and the Hessian of \( f, \nabla^2 f(\vec{x}) \), is positive semi-definite for all \( \vec{x} \in \text{dom}(f) \).

(a) **Restriction to a line.**

Show that a function \( f \) is convex if and only if for all \( \vec{x} \in \text{dom}(f) \) and all \( \vec{v} \), the function \( g: \text{dom}(g) \rightarrow \mathbb{R} \) given by \( g(t) = f(\vec{x} + t\vec{v}) \) is convex for \( \text{dom}(g) = \{ t \in \mathbb{R} \mid \vec{x} + t\vec{v} \in \text{dom}(f) \} \).
(b) **Non-negative weighted sum.**
Show that the non-negative weighted sum of convex functions is convex: i.e. if \( f_1, \ldots, f_n \) are \( n \) convex functions from \( \mathbb{R}^n \) to \( \mathbb{R} \) and \( w_1, \ldots, w_n \in \mathbb{R}_+ \) are \( n \) positive scalars, then the function:
\[
f = \sum_{i=1}^{n} w_i f_i
\]  
(5)
is convex. To make the question easier, you can assume that the functions \( f_1, \ldots, f_n \) are twice-differentiable.

(c) **Point-wise maximum.**
Show that if \( f_1 \) and \( f_2 \) are convex functions then their pointwise maximum \( f \), defined by
\[
f(\vec{x}) = \max(f_1(\vec{x}), f_2(\vec{x})),
\]  
(6)
with \( \text{dom}(f) = \text{dom}(f_1) \cap \text{dom}(f_2) \), is also convex.

2. **Convexity of Constraint Sets**
Let \( f_1, \ldots, f_m, h_1, \ldots, h_p : \mathbb{R}^n \rightarrow \mathbb{R} \) be functions. Let \( S \subseteq \mathbb{R}^n \) be defined as
\[
S = \left\{ \vec{x} \in \mathbb{R}^n \left| \begin{array}{lr}
f_i(\vec{x}) \leq 0 & \forall i = 1, \ldots, m \\
h_j(\vec{x}) = 0 & \forall j = 1, \ldots, p
\end{array} \right. \right\}.
\]  
(7)
Show that if \( f_1, \ldots, f_m \) are convex functions, and \( h_1, \ldots, h_p \) are affine functions, then \( S \) is a convex set.
3. Ridge Regression

Prove that the optimal solution to the ridge regression problem:

$$\min_{\vec{w} \in \mathbb{R}^p} \|X\vec{w} - \vec{y}\|^2_2 + \lambda \|\vec{w}\|^2_2,$$

where $X \in \mathbb{R}^{n \times p}$, $\lambda > 0$ and $\vec{y} \in \mathbb{R}^n$, is given by:

$$\vec{w}^* = (X^T X + \lambda I)^{-1} X^T \vec{y}.$$