Discussion 8

1. Simple Constrained Optimization Problem

Consider the optimization problem

$$\begin{aligned} \min_{x_1, x_2 \in \mathbb{R}} \quad & f(x_1, x_2) && (1) \\ \text{s.t.} \quad & 2x_1 + x_2 \geq 1 && (2) \\ & x_1 + 3x_2 \geq 1 && (3) \\ & x_1 \geq 0,\ x_2 \geq 0 && (4) \end{aligned}$$

(a) Make a sketch of the feasible set.

For each of the following objective functions, give the optimal set or the optimal value.

(b) $f(x_1, x_2) = x_1 + x_2$.

(c) $f(x_1, x_2) = -x_1 - x_2$.

(d) $f(x_1, x_2) = x_1$

(e) $f(x_1, x_2) = \max\{x_1, x_2\}$

(f) $f(x_1, x_2) = x_1^2 + 9x_2^2$

2. Simple Constrained Optimization Problem with Duality

Consider the optimization problem

$$\begin{aligned} \min_{x_1, x_2 \in \mathbb{R}} \quad & f(x_1, x_2) && (5) \\ \text{s.t.} \quad & 2x_1 + x_2 \geq 1 && (6) \\ & x_1 + 3x_2 \geq 1 && (7) \\ & x_1 \geq 0, && (8) \\ & x_2 \geq 0 && (9) \end{aligned}$$

(a) Express the Lagragian of the problem $\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

(b) Show that $\mathcal{L}$ is concave in $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

(c) Express the dual function of the problem, and show that it is concave.

(d) Assume f is convex. Show that $\mathcal{L}$ is convex in (x_1, x_2).

(e) Denoting $\mathcal{X} = \{(x_1, x_2) \mid 2x_1 + x_2 \geq 1,\ x_1 + 3x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0\}$, show that

$$\max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{cases} f(x_1, x_2) & \text{if } (x_1, x_2) \in \mathcal{X} \\ +\infty & \text{otherwise} \end{cases} \quad (10)$$

(f) Conclude that $\min_{(x_1,x_2)\in\mathcal{X}} \max_{\lambda_1\geq 0,\lambda_2\geq 0,\lambda_3\geq 0,\lambda_4\geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \min_{(x_1,x_2)\in\mathcal{X}} f(x_1, x_2)$.

(g) Assuming f is convex, formulate the first order condition on $\mathcal{L}$ (i.e., $\nabla_{x_1,x_2}\mathcal{L}(x_1^\star, x_2^\star, \lambda_1, \lambda_2, \lambda_3, \lambda_5) = 0$) as a function of ∇f and $\lambda_1, \lambda_2, \lambda_3$ and λ_4 to solve:

$$\min_{x_1,x_2} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \tag{11}$$

