EECS 127/227AT Optimization Models in Engineering UC Berkeley Spring 2024

Discussion 8

1. Simple Constrained Optimization Problem

Consider the optimization problem

$$\begin{aligned} \min_{x_1, x_2 \in \mathbb{R}} \quad & f(x_1, x_2) && (1) \\ \text{s.t.} \quad & 2x_1 + x_2 \geq 1 && (2) \\ & x_1 + 3x_2 \geq 1 && (3) \\ & x_1 \geq 0, \; x_2 \geq 0 && (4) \end{aligned}$$

(a) Make a sketch of the feasible set.

Solution: See figure 1.

For each of the following objective functions, give the optimal set or the optimal value.

(b) $f(x_1, x_2) = x_1 + x_2$.

Solution: Using the drawing (figure 2) it seems that the solution is such that $x_1^\star = \frac{2}{5}$ and $x_2^\star = \frac{1}{5}$.

One can verify the optimality of such point using the first order convexity condition:

$$\nabla f \left(\frac{2}{5}, \frac{1}{5} \right)^\top \left((x_1, x_2) - \left(\frac{2}{5}, \frac{1}{5} \right) \right) \geq 0, \quad \forall (x_1, x_2) \in \mathcal{X} \tag{5}$$

Where $\mathcal{X}$ is the feasible set.

It can also be derived using strong duality (see next lecture on duality).

(c) $f(x_1, x_2) = -x_1 - x_2$.

Solution: Here (figure 2) the problem is unbounded below as if $(x_1, x_2) = t(1, 1)$ with $t \geq 0$ then (x_1, x_2) is always feasible and $-2t \to -\infty$ when $t \to \infty$.

(d) $f(x_1, x_2) = x_1$

Solution: The set of solutions is $S = \{\vec{x}, x_1 = 0 \text{ and } x_2 \geq 1\}$ (see figure 2).

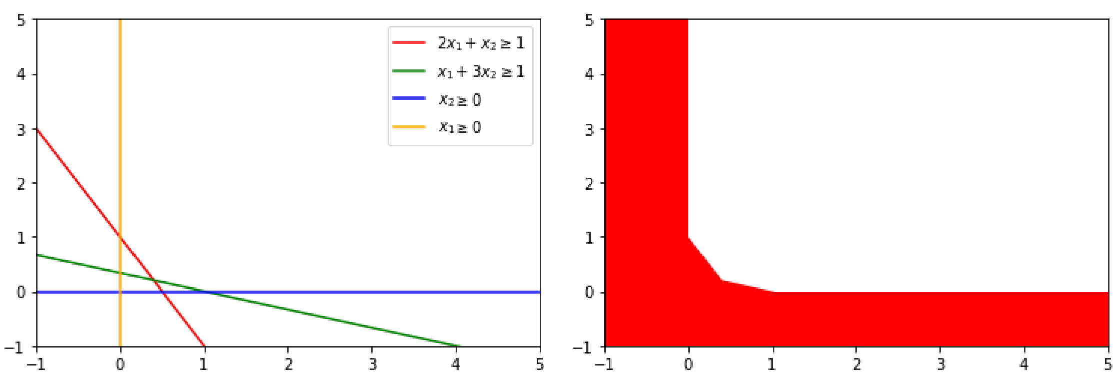
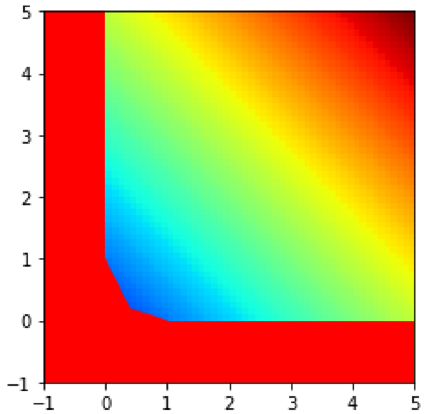
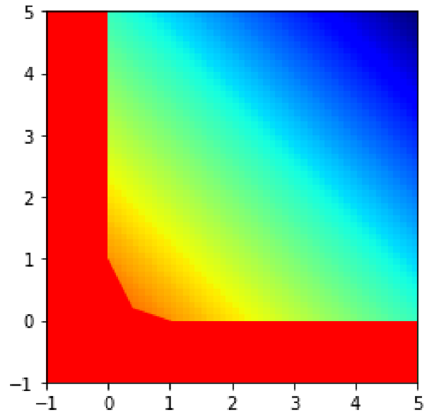


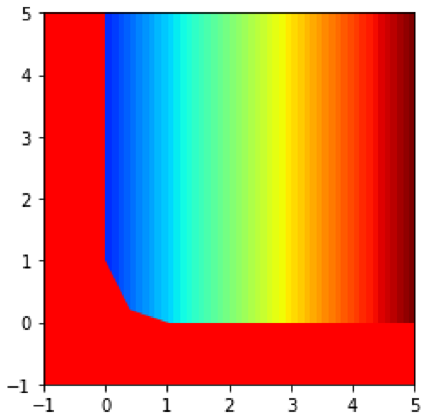
Figure 1: The feasible set is in white on the right figure.



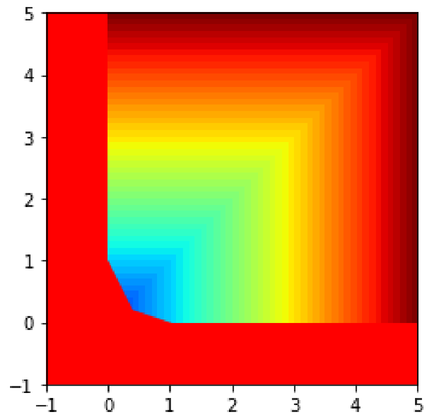
(b)



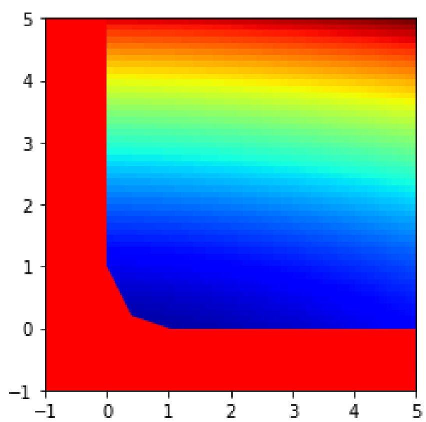
(c)



(d)



(e)



(f)

Figure 2: Solution of (b): $\vec{x}^\star = (\frac{2}{5}, \frac{1}{5})$, (c) is unbounded below, solutions of (d): $\vec{x}^\star = \{(0, x_2) \mid x_2 \geq 1\}$, solution of (e): $\vec{x}^\star = (\frac{1}{4}, \frac{1}{4})$, solution of (f): $\vec{x}^\star = (\frac{2}{5}, \frac{1}{5})$. In red is the infeasible points, then the level sets are shown with colors; blue points are points (x_1, x_2) with the lowest value $f(x_1, x_2)$, red points are the ones with highest value.

(e) $f(x_1, x_2) = \max\{x_1, x_2\}$

Solution: Using the drawing (see figure 2) it seems that the solution is such that:

$$x_1^\star = x_2^\star = \frac{1}{3}. \tag{6}$$

Here, it might be hard to use the first order convexity condition, as the objective function is not differentiable (you can use sub-gradients, but it is beyond the scope of class).

Another technique is to use a slack variable. The problem is equivalent to

$$\begin{aligned} \min_{x_1, x_2, t} \quad & t && (7) \\ \text{s.t.} \quad & t \geq x_1, t \geq x_2 && (8) \\ & 2x_1 + x_2 \geq 1 && (9) \\ & x_1 + 3x_2 \geq 1 && (10) \\ & x_1 \geq 0,\ x_2 \geq 0 && (11) \end{aligned}$$

Here we can use the first order condition to check optimality conditions.

(f) $f(x_1, x_2) = x_1^2 + 9x_2^2$

Solution: Using the drawing (see figure 2) it seems that the solution is such that $x_1^\star = \frac{1}{2}$ and $x_2^\star = \frac{1}{6}$.

It can be verified using the first order convexity condition:

$$\nabla f\left(\frac{1}{2}, \frac{1}{6}\right)^\top \left((x_1, x_2) - \left(\frac{1}{2}, \frac{1}{6}\right)\right) \geq 0, \quad \forall (x_1, x_2) \in \mathcal{X} \tag{12}$$

Where $\mathcal{X}$ is the feasible set.

2. Simple Constrained Optimization Problem with Duality

Consider the optimization problem

$$\begin{aligned} \min_{x_1, x_2 \in \mathbb{R}} \quad & f(x_1, x_2) && (13) \\ \text{s.t.} \quad & 2x_1 + x_2 \geq 1 && (14) \\ & x_1 + 3x_2 \geq 1 && (15) \\ & x_1 \geq 0, && (16) \\ & x_2 \geq 0 && (17) \end{aligned}$$

(a) Express the Lagragian of the problem $\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

Solution:

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = f(x_1, x_2) + \lambda_1(-2x_1 - x_2 + 1) + \lambda_2(1 - x_1 - 3x_2) - \lambda_3 x_1 - \lambda_4 x_2 \tag{18}$$

(b) Show that $\mathcal{L}$ is concave in $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

Solution: $-\mathcal{L}$ is convex in $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ as a affine function of $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. So $\mathcal{L}$ is concave in $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$

(c) Express the dual function of the problem, and show that it is concave.

Solution: $g(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \min_{x_1,x_2} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

We can show that by showing that $-g$ is convex.

$$-g(\vec{\lambda}) = -\min_{x_1,x_2} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \tag{19}$$

$$= \max_{x_1,x_2} -\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \tag{20}$$

When (x_1, x_2) is fixed, the function $-\mathcal{L}$ is linear in $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, therefore convex.

Because the max of convex functions is convex, $-g$ is convex. Therefore g is concave.

(d) Assume f is convex. Show that $\mathcal{L}$ is convex in (x_1, x_2).

Solution: $\mathcal{L}$ is convex in (x_1, x_2) because it is the sum of convex functions.

(e) Denoting $\mathcal{X} = \{(x_1, x_2) \mid 2x_1 + x_2 \geq 1,\ x_1 + 3x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0\}$, show that

$$\max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{cases} f(x_1, x_2) & \text{if } (x_1, x_2) \in \mathcal{X} \\ +\infty & \text{otherwise} \end{cases} \tag{21}$$

Solution: Let's just do it for λ_4:

$$\max_{\lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \max_{\lambda_4 \geq 0} (f(x_1, x_2) + \lambda_1(-2x_1 - x_2 + 1) + \lambda_2(1 - x_1 - 3x_2) - \lambda_3 x_1 - \lambda_4 x_2) \tag{22}$$

$$= f(x_1, x_2) + \lambda_1(-2x_1 - x_2 + 1) + \lambda_2(1 - x_1 - 3x_2) - \lambda_3 x_1 + \max_{\lambda_4 \geq 0} -\lambda_4 x_2 \tag{23}$$

$$\max_{\lambda_4 \geq 0} -\lambda_4 x_2 = \begin{cases} 0 & \text{if } x_2 \geq 0 \\ +\infty & \text{otherwise} \end{cases} \tag{24}$$

One can show the same results for λ_1, λ_2 and λ_3, resulting in:

$$\max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{cases} f(x_1, x_2) & \text{if } (x_1, x_2) \in \mathcal{X} \\ +\infty & \text{otherwise} \end{cases} \tag{25}$$

(f) Conclude that $\min_{(x_1,x_2) \in \mathcal{X}} \max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \min_{(x_1,x_2) \in \mathcal{X}} f(x_1, x_2)$.

Solution:

$$\min_{x_1,x_2} \max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \min_{x_1,x_2} \begin{cases} f(x_1, x_2) & \text{if } (x_1, x_2) \in \mathcal{X} \\ +\infty & \text{otherwise} \end{cases} \tag{26}$$

$$= \min_{(x_1,x_2) \in \mathcal{X}} f(x_1, x_2) \tag{27}$$

(g) Assuming f is convex, formulate the first order condition on $\mathcal{L}$ (i.e., $\nabla_{x_1,x_2}\mathcal{L}(x_1^\star, x_2^\star, \lambda_1, \lambda_2, \lambda_3, \lambda_5) = 0$) as a function of ∇f and $\lambda_1, \lambda_2, \lambda_3$ and λ_4 to solve:

$$\min_{x_1,x_2} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \tag{28}$$

Solution:

$$\nabla_{x_1,x_2}\mathcal{L}(x_1^\star, x_2^\star, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = 0 \tag{29}$$

$$= \nabla_{x_1,x_2} f(x_1^\star, x_2^\star) + \begin{bmatrix} -2\lambda_1 - \lambda_2 - \lambda_3 \\ -\lambda_1 - 3\lambda_2 - \lambda_4 \end{bmatrix} \tag{30}$$

