1. Simple Constrained Optimization Problem

Consider the optimization problem

\[
\begin{align*}
\min_{x_1, x_2 \in \mathbb{R}} & \quad f(x_1, x_2) \\
\text{s.t.} & \quad 2x_1 + x_2 \geq 1 \\
& \quad x_1 + 3x_2 \geq 1 \\
& \quad x_1 \geq 0, \quad x_2 \geq 0
\end{align*}
\]  

(a) Make a sketch of the feasible set.

**Solution:** See figure 1.

For each of the following objective functions, give the optimal set or the optimal value.

(b) \(f(x_1, x_2) = x_1 + x_2\).

**Solution:** Using the drawing (figure 2) it seems that the solution is such that \(x_1^* = \frac{2}{5}\) and \(x_2^* = \frac{1}{5}\).

One can verify the optimality of such point using the first order convexity condition:

\[
\nabla f \left( \frac{2}{5}, \frac{1}{5} \right) \top \left( (x_1, x_2) - \left( \frac{2}{5}, \frac{1}{5} \right) \right) \geq 0, \quad \forall (x_1, x_2) \in \mathcal{X}
\]

Where \(\mathcal{X}\) is the feasible set.

It can also be derived using strong duality (see next lecture on duality).

(c) \(f(x_1, x_2) = -x_1 - x_2\).

**Solution:** Here (figure 2) the problem is unbounded below as if \((x_1, x_2) = t(1, 1)\) with \(t \geq 0\) then \((x_1, x_2)\) is always feasible and \(-2t \to -\infty\) when \(t \to \infty\).

(d) \(f(x_1, x_2) = x_1\)

**Solution:** The set of solutions is \(S = \{x, x_1 = 0 \text{ and } x_2 \geq 1\}\) (see figure 2).

![Figure 1: The feasible set is in white on the right figure.](image)
Figure 2: Solution of (b): $\vec{x}^* = (\frac{2}{5}, \frac{1}{5})$. (c) is unbounded below, solutions of (d): $\vec{x}^* = \{(0, x_2) \mid x_2 \geq 1\}$, solution of (e): $\vec{x}^* = (\frac{1}{4}, \frac{1}{4})$, solution of (f): $\vec{x}^* = (\frac{2}{5}, \frac{1}{5})$. In red is the infeasible points, then the level sets are shown with colors; blue points are points $(x_1, x_2)$ with the lowest value $f(x_1, x_2)$, red points are the ones with highest value.
(e) $f(x_1, x_2) = \max\{x_1, x_2\}$

**Solution:** Using the drawing (see figure 2) it seems that the solution is such that:

$$x_1^* = x_2^* = \frac{1}{3}.$$  \hfill (6)

Here, it might be hard to use the first order convexity condition, as the objective function is not differentiable (you can use sub-gradients, but it is beyond the scope of class).

Another technique is to use a slack variable. The problem is equivalent to

$$\min_{x_1, x_2, t} t$$

s.t.  \hfill (7)

$$t \geq x_1, t \geq x_2 \quad \text{s.t.}$$

$$2x_1 + x_2 \geq 1$$ \hfill (8)

$$x_1 + 3x_2 \geq 1 \quad \text{s.t.}$$ \hfill (9)

$$x_1 \geq 0, \ x_2 \geq 0$$ \hfill (10)

Here we can use the first order condition to check optimality conditions.

(f) $f(x_1, x_2) = x_1^2 + 9x_2^2$

**Solution:** Using the drawing (see figure 2) it seems that the solution is such that $x_1^* = \frac{1}{2}$ and $x_2^* = \frac{1}{6}$.

It can be verified using the first order convexity condition:

$$\nabla f \left( \frac{1}{2}, \frac{1}{6} \right)^T \left( (x_1, x_2) - \left( \frac{1}{2}, \frac{1}{6} \right) \right) \geq 0, \ \forall (x_1, x_2) \in \mathcal{X}'$$ \hfill (12)

Where $\mathcal{X}'$ is the feasible set.

2. **Simple Constrained Optimization Problem with Duality**

Consider the optimization problem

$$\min_{x_1, x_2 \in \mathbb{R}} f(x_1, x_2)$$ \hfill (13)

s.t.  \hfill (14)

$$2x_1 + x_2 \geq 1$$

$$x_1 + 3x_2 \geq 1$$

$$x_1 \geq 0, \ x_2 \geq 0$$

(a) Express the Lagrangian of the problem $\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

**Solution:**

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = f(x_1, x_2) + \lambda_1(-2x_1 - x_2 + 1) + \lambda_2(1 - x_1 - 3x_2) - \lambda_3 x_1 - \lambda_4 x_2$$ \hfill (18)

(b) Show that $\mathcal{L}$ is concave in $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

**Solution:** $-\mathcal{L}$ is convex in $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ as a affine function of $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. So $\mathcal{L}$ is concave in $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.
(c) Express the dual function of the problem, and show that it is concave.

**Solution:** \( g(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \min_{x_1, x_2} L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4). \)

We can show that by showing that \(-g\) is convex.

\[
-g(\bar{x}) = - \min_{x_1, x_2} L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \max_{x_1, x_2} -L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \tag{19}
\]

When \((x_1, x_2)\) is fixed, the function \(-L\) is linear in \((\lambda_1, \lambda_2, \lambda_3, \lambda_4)\), therefore convex. Because the max of convex functions is convex, \(-g\) is convex. Therefore \(g\) is concave.

(d) Assume \(f\) is convex. Show that \(L\) is convex in \((x_1, x_2)\).

**Solution:** \(L\) is convex in \((x_1, x_2)\) because it is the sum of convex functions.

(e) Denoting \(\mathcal{X} = \{(x_1, x_2) \mid 2x_1 + x_2 \geq 1, \ x_1 + 3x_2 \geq 1, \ x_1 \geq 0, \ x_2 \geq 0\}\), show that

\[
\max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{cases} f(x_1, x_2) & \text{if } (x_1, x_2) \in \mathcal{X} \\ +\infty & \text{otherwise} \end{cases} \tag{21}
\]

**Solution:** Let’s just do it for \(\lambda_4\):

\[
\max_{\lambda_4 \geq 0} L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \max_{\lambda_4 \geq 0} (f(x_1, x_2) + \lambda_1(-2x_1 - x_2 + 1) + \lambda_2(1 - x_1 - 3x_2) - \lambda_3x_1 - \lambda_4x_2) \tag{22}
\]

\[
= f(x_1, x_2) + \lambda_1(-2x_1 - x_2 + 1) + \lambda_2(1 - x_1 - 3x_2) - \lambda_3x_1 + \max_{\lambda_4 \geq 0} -\lambda_4x_2 \tag{23}
\]

\[
\max_{\lambda_4 \geq 0} -\lambda_4x_2 = \begin{cases} 0 & \text{if } x_2 \geq 0 \\ +\infty & \text{otherwise} \end{cases} \tag{24}
\]

One can show the same results for \(\lambda_1, \lambda_2\) and \(\lambda_3\), resulting in:

\[
\max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{cases} f(x_1, x_2) & \text{if } (x_1, x_2) \in \mathcal{X} \\ +\infty & \text{otherwise} \end{cases} \tag{25}
\]

(f) Conclude that \(\min_{(x_1, x_2) \in \mathcal{X}} \max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \min_{(x_1, x_2) \in \mathcal{X}} f(x_1, x_2). \)

**Solution:**

\[
\min_{x_1, x_2} \max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \min_{x_1, x_2} \begin{cases} f(x_1, x_2) & \text{if } (x_1, x_2) \in \mathcal{X} \\ +\infty & \text{otherwise} \end{cases} \tag{26}
\]

\[
= \min_{(x_1, x_2) \in \mathcal{X}} f(x_1, x_2) \tag{27}
\]
(g) Assuming $f$ is convex, formulate the first order condition on $L$ (i.e., $\nabla_{x_1,x_2} L(x_1^*, x_2^*, \lambda_1, \lambda_2, \lambda_3, \lambda_5) = 0$) as a function of $\nabla f$ and $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$ to solve:

$$\min_{x_1,x_2} L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$$  \hspace{1cm} (28)

**Solution:**

$$\nabla_{x_1,x_2} L(x_1^*, x_2^*, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = 0$$  \hspace{1cm} (29)

$$= \nabla_{x_1,x_2} f(x_1^*, x_2^*) + \begin{bmatrix} -2\lambda_1 - \lambda_2 - \lambda_3 \\ -\lambda_1 - 3\lambda_2 - \lambda_4 \end{bmatrix}$$  \hspace{1cm} (30)