Discussion 9

1. An optimization problem

Consider the primal optimization problem,

$$p^\star = \min_{x \in \mathbb{R}^2} \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \quad (1)$$

$$\text{s.t.} \quad x_1 \geq 0 \quad (2)$$

$$x_1 + x_2 \geq 2. \quad (3)$$

First we solve the primal problem directly.

(a) Sketch the feasible region and argue that $x^\star = (1, 1)$ and $p^\star = 1$.

(b) The *critical points* of an optimization problem are points where the gradient is 0 or undefined, and also points which are on the boundary of the constraint set.

Compute the value of the objective function at its critical points and find $p^\star$ and $x^\star$.

(c) Next we solve the problem with the help of the dual. First, find the Lagrangian $\mathcal{L}(x, \lambda)$.

(d) Formulate the dual problem.

(e) Solve the dual problem to find $d^\star$ and $\lambda^\star$.

(f) Does strong duality hold?

(g) Find $p^\star$ and $x^\star$.

(h) Finally we use KKT conditions to find $x^\star, \lambda^\star$. First, Write down the KKT conditions and find $\tilde{x}$ and $\tilde{\lambda}$ that satisfy it.

(i) Argue why the optimal primal and dual solutions are given by $x^\star = \tilde{x}$ and $\lambda^\star = \tilde{\lambda}$.

2. Lagrangian Dual of a QP

Consider the general form of a convex quadratic program, with $Q \succ 0$:

$$\min_{\vec{x}} \quad \frac{1}{2}\vec{x}^\top Q\vec{x} \tag{4}$$

$$\text{s.t.} \quad A\vec{x} \leq \vec{b} \tag{5}$$

(a) Write the Lagrangian function $\mathcal{L}(\vec{x}, \vec{\lambda})$.

(b) Write the Lagrangian dual function, $g(\vec{\lambda})$.

(c) Show that the Lagrangian dual problem is convex by writing it in standard QP form. Is the Lagrangian dual problem convex in general?