Discussion 10

1. Magic with constraints

In this question, we will represent a problem in two different ways and show that strong duality holds in one case but doesn't hold in the other.

Let

$$f_0(x) \doteq \begin{cases} x^3 - 3x^2 + 4, & x \geq 0 \\ -x^3 - 3x^2 + 4, & x < 0. \end{cases} \tag{1}$$

(a) Consider the minimization problem

$$p^* = \inf_{x \in \mathbb{R}} \quad f_0(x) \tag{2}$$

$$\text{s.t.} \quad -1 \leq x, \;\; x \leq 1. \tag{3}$$

i. Show that $p^* = 2$ and the the set of optimizers $x \in \mathcal{X}^*$ is $\mathcal{X}^* = \{-1, 1\}$ by examining the "critical" points, i.e., points where the gradient is zero, points on the boundaries, and $\pm\infty$.

ii. Show that the dual problem can be represented as

$$d^* = \sup_{\lambda_1, \lambda_2 \geq 0} g(\vec{\lambda}), \tag{4}$$

where

$$g(\vec{\lambda}) = \min \left\{ g_1(\vec{\lambda}), g_2(\vec{\lambda}) \right\}, \tag{5}$$

with

$$g_1(\vec{\lambda}) = \inf_{x \geq 0} x^3 - 3x^2 + 4 - \lambda_1(x+1) + \lambda_2(x-1) \tag{6}$$

$$g_2(\vec{\lambda}) = \inf_{x < 0} -x^3 - 3x^2 + 4 - \lambda_1(x+1) + \lambda_2(x-1). \tag{7}$$

iii. Next, show that

$$g_1(\vec{\lambda}) \leq -3\lambda_1 + \lambda_2 \tag{8}$$

$$g_2(\vec{\lambda}) \leq \lambda_1 - 3\lambda_2. \tag{9}$$

Use this to show that $g(\vec{\lambda}) \leq 0$ for all $\lambda_1, \lambda_2 \geq 0$.

iv. Show that $g(\vec{0}) = 0$ and conclude that $d^* = 0$.

v. Does strong duality hold?

(b) Now, consider a problem equivalent to the minimization in (2):

$$p^* = \inf_{x \in \mathbb{R}} \quad f_0(x) \tag{10}$$

$$\text{s.t.} \quad x^2 \leq 1. \tag{11}$$

Observe that $p^* = 2$ and the set of optimizers $x \in \mathcal{X}^*$ is $\mathcal{X}^* = \{-1, 1\}$, since this problem is equivalent to the one in part (a).

i. Show that the dual problem can be represented as

$$d^* = \sup_{\lambda \geq 0} g(\lambda), \tag{12}$$

where

$$g(\lambda) = \min\{g_1(\lambda), g_2(\lambda)\}, \tag{13}$$

with

$$g_1(\lambda) = \inf_{x \geq 0} x^3 - 3x^2 + 4 + \lambda(x^2 - 1) \tag{14}$$

$$g_2(\lambda) = \inf_{x < 0} -x^3 - 3x^2 + 4 + \lambda(x^2 - 1). \tag{15}$$

ii. Show that $g_1(\lambda) = g_2(\lambda) = \begin{cases} 4 - \lambda, & \lambda \geq 3 \\ -\frac{4}{27}(3 - \lambda)^3 + 4 - \lambda, & 0 \leq \lambda < 3. \end{cases}$.

iii. Conclude that $d^* = 2$ and the optimal $\lambda = \frac{3}{2}$.

iv. Does strong duality hold?

2. Complementary Slackness

Consider the problem:

$$p^\star = \min_{x \in \mathbb{R}} \quad x^2 \tag{16}$$

$$\text{s.t.} \quad x \geq 1,\ x \leq 2. \tag{17}$$

(a) Does Slater's condition hold? Is the problem convex? Does strong duality hold?

(b) Find the Lagrangian $\mathcal{L}(x, \lambda_1, \lambda_2)$.

(c) Find the dual function $g(\lambda_1, \lambda_2)$ so that the dual problem is given by,

$$d^\star = \max_{\lambda_1, \lambda_2 \in \mathbb{R}_+} g(\lambda_1, \lambda_2). \tag{18}$$

(d) Solve the dual problem in (18) for $d^\star$.

(e) Solve for $x^\star, \lambda_1^\star, \lambda_2^\star$ that satisfy KKT conditions.

(f) Can you spot a connection between the values of $\lambda_1^\star, \lambda_2^\star$ in relation to whether the corresponding inequality constraints are strict or not at the optimal $x^\star$?