Discussion 11

We might not cover all the problems on the worksheet, as discussion worksheets are not always designed to be finished within an hour. However, this is totally fine since they are deliberately made slightly longer so they can serve as resources you can use to practice, reinforce, and build upon concepts covered in lectures and homework.

1. A Linear Program

Let $A \in \mathbb{R}^{m \times n}$, $\vec{y} \in \mathbb{R}^m$ and $\mu > 0$. First, consider the following problem:

$$p^\star = \min_{\vec{x}} \|A\vec{x} - \vec{y}\|_1 . \tag{1}$$

For $i \in \{1, \ldots, m\}$, we denote by $\vec{a}_i^\top$ the i-th row of A, so that $A = \begin{bmatrix} \vec{a}_1^\top \\ \vdots \\ \vec{a}_m^\top \end{bmatrix}$.

(a) Express the problem as an LP.

(b) Show that a dual to the problem can be written as

$$d^\star = \max_{\vec{u}} -\vec{u}^\top \vec{y} \; : \; A^\top \vec{u} = 0, \quad \|\vec{u}\|_\infty \leq 1. \tag{2}$$

HINT: Use the fact that, for any vector z:

$$\max_{\vec{u} \,:\, \|\vec{u}\|_1 \leq 1} \vec{u}^\top \vec{z} = \|\vec{z}\|_\infty , \quad \max_{\vec{u} \,:\, \|\vec{u}\|_\infty \leq 1} \vec{u}^\top \vec{z} = \|\vec{z}\|_1 . \tag{3}$$

Additionally, you may use the following fact: If the primal problem is expressed as $p^\star = \min_{\vec{x}} \max_{\vec{y}} L(\vec{x}, \vec{y})$ *(which is* $p^\star = \min_{\vec{x}} f_0(x)$ *for* $f_0(x) = \max_{\vec{y}} L(\vec{x}, \vec{y})$*), then the dual problem can be obtained by swapping the max and min:* $d^\star = \max_{\vec{y}} \min_{\vec{x}} L(\vec{x}, \vec{y})$.

Now, consider the following more complicated problem involving both the ℓ_1 and ℓ_∞ norms:

$$p^\star = \min_{\vec{x}} \; \|A\vec{x} - \vec{y}\|_1 + \mu \|\vec{x}\|_\infty . \tag{4}$$

(c) Express the problem as an LP.

(d) Show that a dual to the problem can be written as

$$\begin{aligned} d^\star = \max_{\vec{u}} \; & -\vec{u}^\top \vec{y} \\ \text{s.t.} \; & \|\vec{u}\|_\infty \leq 1, \;\; \left\|A^\top \vec{u}\right\|_1 \leq \mu. \end{aligned}$$

HINT: Use the fact that, for any vector z:

$$\max_{\vec{u} \,:\, \|\vec{u}\|_1 \leq 1} \vec{u}^\top \vec{z} = \|\vec{z}\|_\infty , \quad \max_{\vec{u} \,:\, \|\vec{u}\|_\infty \leq 1} \vec{u}^\top \vec{z} = \|\vec{z}\|_1 . \tag{5}$$

Additionally, you may use the following fact: If the primal problem is expressed as $p^\star = \min_{\vec{x}} \max_{\vec{y}} L(\vec{x}, \vec{y})$ *(which is* $p^\star = \min_{\vec{x}} f_0(x)$ *for* $f_0(x) = \max_{\vec{y}} L(\vec{x}, \vec{y})$*), then the dual problem can be obtained by swapping the max and min:* $d^\star = \max_{\vec{y}} \min_{\vec{x}} L(\vec{x}, \vec{y})$.

2. A review of standard problem formulations

In this question, we review conceptually the standard forms of various problems and the assertions we can (and cannot!) make about each.

(a) ***Linear programming (LP).***

i. Write the most general form of a linear program (LP) and list its defining attributes.

ii. Under what conditions is an LP convex?

(b) ***Quadratic programming (QP).***

i. Write the most general form of a quadratic program (QP) and list its defining attributes.

ii. Under what conditions is a QP convex?

(c) ***Quadratically-constrained quadratic programming (QCQP).***

i. Write the most general form of a quadratically-constrained quadratic program (QCQP) and list its defining attributes.

ii. Under what conditions is a QCQP convex?

(d) ***Second-order cone programming (SOCP).***

i. Write the most general form of a second-order cone program (SOCP) and list its defining attributes.

ii. Under what conditions is an SOCP convex?

(e) ***Relationships.*** Recall that

$$\text{LP} \subset \text{QP}_{\text{convex}} \subset \text{QCQP}_{\text{convex}} \subset \text{SOCP} \subset \{\text{all convex programs}\}, \tag{6}$$

where LP denotes the set of all linear programs, $\text{QP}_{\text{convex}}$ denotes the set of all convex quadratic programs, etc. Which of these problems can be solved most efficiently? Why are these categorizations useful?

