1. Robust Linear Programming

In this problem we will consider a version of linear programming under uncertainty.

(a) Consider vector $\vec{x} \in \mathbb{R}^n$. Recall that $\vec{x}^\top \vec{y} \leq \|\vec{x}\|_1$ for all $\vec{y}$ such that $\|\vec{y}\|_\infty \leq 1$. Further this inequality is tight, since it holds with equality for $\vec{y} = \text{sign}(\vec{x})$. You saw this in the previous homework, just remind yourself of the solution, no need to turn anything in.

Let us focus now on a LP in standard form:

$$\min \quad \vec{c}^\top \vec{x}$$
$$\text{s.t.} \quad \vec{a}_i^\top \vec{x} \leq b_i, \quad i = 1, \ldots, m.$$  

Consider the set of linear inequalities in (2). Suppose you don’t know the vectors $\vec{a}_i$ exactly. Instead you are given nominal values $\vec{\hat{a}}_i$, and you know that the actual vectors satisfy $\|\vec{a}_i - \vec{\hat{a}}_i\|_\infty \leq \rho$ for a given $\rho > 0$. In other words, the actual components $a_{ij}$ can be anywhere in the intervals $[\hat{a}_{ij} - \rho, \hat{a}_{ij} + \rho]$. Or equivalently, each vector $\vec{a}_i$ can lie anywhere in a hypercube with corners $\vec{\hat{a}}_i + \vec{v}$ where $\vec{v} \in \{-\rho, \rho\}^n$.

We desire that the set of inequalities that constrain problem (2) be satisfied for all possible values of $\vec{a}_i$; i.e., we replace these with the constraints

$$\vec{\hat{a}}_i^\top \vec{x} \leq b_i \quad \forall \vec{a}_i \in \{ \vec{\hat{a}}_i + \vec{v} \mid \|\vec{v}\|_\infty \leq \rho \} \quad i = 1, \ldots, m.$$  

Note that the above defines an infinite number of constraints (of the form $\vec{\hat{a}}_i^\top \vec{x} + \vec{v}^\top \vec{x} \leq b_i$, $\forall \vec{v}$ satisfying $\|\vec{v}\|_\infty \leq \rho$, $i = 1, 2, \ldots, m$).

(b) Argue why for our LP we can replace the infinite set of constraints as above to a finite set of $2^n m$ constraints of the form,

$$\vec{\hat{a}}_i^\top \vec{x} + \rho \|\vec{x}\|_1 \leq b_i, \quad i = 1, \ldots, m.$$  

HINT: What do you know about the optimal solutions of LPs?

(c) Use result from part (a) to show that the constraint set in Equation (3) is in fact equivalent to the much more compact set of $m$ nonlinear inequalities

$$\vec{\hat{a}}_i^\top \vec{x} + \rho \|\vec{x}\|_1 \leq b_i, \quad i = 1, \ldots, m.$$  

We now would like to formulate the LP with uncertainty introduced. We are therefore interested in situations where the vectors \( \vec{a}_i \) are uncertain, but satisfy bounds \( \| \vec{a}_i - \vec{\hat{a}}_i \|_\infty \leq \rho \) for given \( \vec{\hat{a}}_i \) and \( \rho \).

We want to minimize \( \vec{c}^\top \vec{x} \) subject to the constraint that the inequalities \( \vec{a}_i^\top \vec{x} \leq b_i \) are satisfied for all possible values of \( \vec{a}_i \).

We call this a robust LP:

\[
\min_{\vec{x}} \quad \vec{c}^\top \vec{x} \\
\text{s.t.} \quad \vec{a}_i^\top \vec{x} \leq b_i, \quad \forall \vec{a}_i \in \{ \vec{\hat{a}}_i + \vec{v} | \| \vec{v} \|_\infty \leq \rho \} \quad i = 1, ..., m.
\]

(d) Using the result from part (c), express the above optimization problem as an LP.

2. A review of standard problem formulations

In this question, we review conceptually the standard forms of various problems and the assertions we can (and cannot!) make about each.

(a) **Linear programming (LP).**

    (a) Write the most general form of a linear program (LP) and list its defining attributes.
(b) Under what conditions is an LP convex?

(b) **Quadratic programming (QP).**
(a) Write the most general form of a quadratic program (QP) and list its defining attributes.

(b) Under what conditions is a QP convex?

(c) **Quadratically-constrained quadratic programming (QCQP).**
(a) Write the most general form of a quadratically-constrained quadratic program (QCQP) and list its defining attributes.

(b) Under what conditions is a QCQP convex?

(d) **Second-order cone programming (SOCP).**
(a) Write the most general form of a second-order cone program (SOCP) and list its defining attributes.

(b) Under what conditions is an SOCP convex?

(e) **Relationships.** Recall that

\[ LP \subset QP_{\text{convex}} \subset QCQP_{\text{convex}} \subset SOCP \subset \{\text{all convex programs}\}, \]

where \( LP \) denotes the set of all linear programs, \( QP_{\text{convex}} \) denotes the set of all convex quadratic programs, etc. Which of these problems can be solved most efficiently? Why are these categorizations useful?