Discussion 14

1. Newton's Method

Consider the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = x^4. \tag{1}$$

(a) **Find the optimal value $x^\star = \operatorname{argmin}_x f(x)$.**

Solution: Note that $f(x) = x^4 \geq 0$ and is equal to 0 if and only if $x = 0$. Thus $x^\star = 0$.

(b) Now, we analyze the performance of Newton's method on this problem. Starting from x_0, for $k \geq 0$ we take Newton steps of the form

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}. \tag{2}$$

Find the minimum number of Newton steps that are required to be within a distance of $\epsilon > 0$ from optimum $x^\star$. You may use the optimum value of $x^\star$ from part (a). Formally find $k^\star \in \mathbb{N}$ (where $\mathbb{N}$ is the set of positive integers) which is the smallest k for which $|x_k - x^\star| \leq \epsilon$, i.e.,

$$k^\star = \min_{k \in \mathbb{N} : |x_k - x^\star| \leq \epsilon} k. \tag{3}$$

Assume that $x_0 > \epsilon > 0$. Your answer should be in terms of ϵ and x_0.

Solution: First we calculate the derivatives of $f(x)$.

$$f'(x) = 4x^3 \tag{4}$$
$$f''(x) = 12x^2. \tag{5}$$

For $k \geq 0$ we have,

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)} \tag{6}$$
$$= x_k - \frac{4x_k^3}{12x_k^2} \tag{7}$$
$$= x_k - \frac{x_k}{3} \tag{8}$$
$$= \frac{2}{3} x_k. \tag{9}$$

Thus,

$$x_k = \left(\frac{2}{3}\right)^k x_0. \tag{10}$$

Since $x^\star = 0$,

$$|x_k - x^\star| \leq \epsilon \tag{11}$$
$$\implies \left(\frac{2}{3}\right)^k |x_0| \leq \epsilon \tag{12}$$
$$\implies \left(\frac{2}{3}\right)^k \leq \frac{\epsilon}{|x_0|} \tag{13}$$

$$\implies k\log(\frac{2}{3}) \leq \log(\left(\frac{\epsilon}{|x_0|}\right)) \tag{14}$$

$$\implies k \geq \frac{\log(\frac{\epsilon}{|x_0|})}{\log(\frac{2}{3})}. \tag{15}$$

We switched the sign of inequality since $\log(\frac{2}{3}) < 0$. The smallest natural number $k^\star$ for which this occurs is,

$$k^\star = \left\lceil \frac{\log(\frac{\epsilon}{|x_0|})}{\log(\frac{2}{3})} \right\rceil = \left\lceil \frac{\log(\frac{\epsilon}{x_0})}{\log(\frac{2}{3})} \right\rceil. \tag{16}$$

2. **Sphere Enclosure**

Let B_i, $i = 1, \ldots, m$, be m Euclidean balls in $\mathbb{R}^n$, with centers $\vec{x}_i$, and radii $\rho_i \geq 0$. We wish to find a ball B with center $\vec{c} \in \mathbb{R}^n$ of minimum radius $r \geq 0$ that contains all the B_i, $i = 1, \ldots, m$. Cast this problem as an SOCP.

Solution: Let $\vec{c} \in \mathbb{R}^n$ and $r \geq 0$ denote the center and radius of the enclosing ball B, respectively. We express the given balls B_i as

$$B_i = \{\vec{x} : \vec{x} = \vec{x}_i + \vec{\delta}_i, \ \left\|\vec{\delta}_i\right\|_2 \leq \rho_i\}, \quad i = 1, \ldots, m. \tag{17}$$

We have that $B_i \subseteq B$ if and only if

$$\max_{\vec{x} \in B_i} \|\vec{x} - \vec{c}\|_2 \leq r. \tag{18}$$

Note that

$$\max_{\vec{x} \in B_i} \|\vec{x} - \vec{c}\|_2 = \max_{\left\|\vec{\delta}_i\right\|_2 \leq \rho_i} \left\|\vec{x}_i - \vec{c} + \vec{\delta}_i\right\|_2 = \|\vec{x}_i - \vec{c}\|_2 + \rho_i. \tag{19}$$

The last step follows by choosing $\vec{\delta}_i$ in the direction of $\vec{x}_i - \vec{c}$. The problem is then cast as the following SOCP

$$\min_{\vec{c}, r} \quad r \tag{20}$$

$$\text{s.t.} \quad \|\vec{x}_i - \vec{c}\|_2 + \rho_i \leq r, \qquad i = 1, \ldots, m. \tag{21}$$