Self grades are due at 11 PM on January 26, 2024.

1. **Course Setup**

   Please complete the following steps to get access to all course resources.

   (a) Visit the course website at `http://eecs127.github.io/` and familiarize yourself with the syllabus.

   (b) Verify that you can access the class Ed site at `https://edstem.org/us/courses/52173`.

   (c) Verify that you can access the class Gradescope site at `https://www.gradescope.com/courses/699270`.

   (d) When are self grades due for this homework? In general, when are self grades due? Where are the self-grade assignments?

   Solution: Self grades for this homework are due January 26, 2024 at 11 PM PST. In general, self grades are due one week following initial homework submission. Self-grade assignments are on Gradescope and are also accessible using the course website.

   (e) How many homework drops do you get? Are there exceptions?

   Solution: Your two lowest homework grades will be dropped. There are no exceptions, and late work will not be accepted, though you may ask for extensions using the form on the course website.

2. What Prerequisites Have You Taken?

The prerequisites for this course are

- EECS 16A & 16B (Designing Information Devices and Systems I & II) **OR** MATH 54 (Linear Algebra & Differential Equations),
- CS 70 (Discrete Mathematics & Probability Theory), and
- MATH 53 (Multivariable Calculus).

Please fill out the following Google form: `https://forms.gle/nnQrTC2EjdEAAkP49` to tell us which of these courses, or their equivalents, you have taken. If you are unsure of course material overlap, please refer to the EECS 16A, EECS 16B, and CS 70 websites (`https://www.eecs16a.org/`, `https://www.eecs16b.org/`, and `http://www.sp22.eecs70.org/`, respectively) and the MATH 53 textbook (*Multivariable Calculus* by James Stewart). **For the response to this question, write the secret word revealed at the end of the form.**

The course material this semester will rely on knowledge from these prerequisite courses. If you feel shaky on this material, please use the first week to reacquaint yourself with it. We expect you to handle this review on your own; we will not prioritize questions about prerequisite material in office hours.

Solution: The secret word is: `MooncoW`

3. Orthogonality

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ be two linearly independent unit-norm vectors; that is, $\|\vec{x}\|_2 = \|\vec{y}\|_2 = 1$.

(a) Show that the vectors $\vec{u} = \vec{x} - \vec{y}$ and $\vec{v} = \vec{x} + \vec{y}$ are orthogonal.

Solution: Orthogonal means dot product is 0. When x, y are both unit-norm, we have

$$(\vec{x} - \vec{y})^\top (\vec{x} + \vec{y}) = \vec{x}^\top \vec{x} - \vec{y}^\top \vec{y} - \vec{y}^\top \vec{x} + \vec{x}^\top \vec{y} = \vec{x}^\top \vec{x} - \vec{y}^\top \vec{y} = \|\vec{x}\|_2^2 - \|\vec{y}\|_2^2 = 1 - 1 = 0. \quad (1)$$

(b) Find an orthonormal basis for $\operatorname{span}(\vec{x}, \vec{y})$, the subspace spanned by $\vec{x}$ and $\vec{y}$.

Solution: Since $\vec{x}$ and $\vec{y}$ are linearly independent, we have $\vec{x} \neq \vec{y}$ and $\vec{x} \neq -\vec{y}$ (since they are both unit norm). Thus $\vec{u}$ and $\vec{v}$ are nonzero.

Motivated by the first part that asks us to show $\vec{u}$ and $\vec{v}$ are orthogonal, we first see that $\vec{u}, \vec{v}$ form an orthogonal basis for $\operatorname{span}(\vec{u}, \vec{v})$.

The subspace spanned by $\vec{x}, \vec{y}$ is $S_1 = \operatorname{span}(\vec{x}, \vec{y})$, We want to check if $S_2 = \operatorname{span}(\vec{u}, \vec{v})$ is the same set as S_1.

If this is true then $\vec{u}$ and $\vec{v}$ are orthogonal basis vectors for $\operatorname{span}(\vec{x}, \vec{y})$.

First we show $S_1 \subseteq S_2$ by checking that $\vec{z} \in S_1 \implies \vec{z} \in S_2$.

We can express any vector $\vec{z} \in \operatorname{span}(\vec{x}, \vec{y})$ as $\vec{z} = \lambda \vec{x} + \mu \vec{y}$, for some $\lambda, \mu \in \mathbb{R}$. We have $\vec{z} = \alpha \vec{u} + \beta \vec{v}$, where

$$\alpha = \frac{\lambda - \mu}{2}, \quad \beta = \frac{\lambda + \mu}{2}. \quad (2)$$

Hence $\vec{z} \in \operatorname{span}(\vec{u}, \vec{v})$. The converse is also true for similar reasons.

We can find orthonormal basis vectors by dividing each orthogonal basis vector by its norm. The desired orthonormal basis is thus given by $((\vec{x} - \vec{y}) / \|\vec{x} - \vec{y}\|_2, (\vec{x} + \vec{y}) / \|\vec{x} + \vec{y}\|_2)$.

We could have also gotten that $\operatorname{span}(\vec{x}, \vec{y}) = \operatorname{span}(\vec{u}, \vec{v})$ in a slightly faster way by noting that

$$\vec{u} = \vec{x} - \vec{y}, \quad \vec{v} = \vec{x} + \vec{y}, \quad \vec{x} = \frac{\vec{u} + \vec{v}}{2}, \quad \vec{y} = \frac{\vec{v} - \vec{u}}{2} \quad (3)$$

so linear combinations of $\vec{u}$ and $\vec{v}$ are linear combinations of $\vec{x}$ and $\vec{y}$, and vice versa. Thus the spans are the same, e.g., $\operatorname{span}(\vec{x}, \vec{y}) = \operatorname{span}(\vec{u}, \vec{v})$. And the solution proceeds from there in the same way.

One can get this same solution in a quicker way by "dimension-counting" — note that since $\vec{u}, \vec{v}$ are orthogonal, they are linearly independent, so $\operatorname{span}(\vec{u}, \vec{v})$ is a 2-dimensional subspace of $\mathbb{R}^n$. In particular, since $\vec{u}$ and $\vec{v}$ are themselves contained in $\operatorname{span}(\vec{x}, \vec{y})$, we have $\operatorname{span}(\vec{u}, \vec{v})$ is a 2-dimensional subspace of $\operatorname{span}(\vec{x}, \vec{y})$. But since $\operatorname{span}(\vec{x}, \vec{y})$ is 2-dimensional itself, the only 2-dimensional subspace it contains is itself, and so $\operatorname{span}(\vec{u}, \vec{v})$ must equal $\operatorname{span}(\vec{x}, \vec{y})$.

Lastly, we present a completely alternate solution: we could use Gram-Schmidt to construct an orthonormal basis for $\operatorname{span}(\vec{x}, \vec{y})$. This is the normal tool to construct orthonormal sets from linearly independent vectors, but ignores the observation in part (a). The Gram-Schmidt algorithm run on $(\vec{x}, \vec{y})$ produces vectors $(\vec{z}_1, \vec{z}_2)$ defined as

$$\vec{z}_1 = \frac{\vec{x}}{\|\vec{x}\|_2} = \vec{x}, \qquad \vec{z}_2 = \frac{\vec{y} - (\vec{y}^\top \vec{x})\vec{x}}{\|\vec{y} - (\vec{y}^\top \vec{x})\vec{x}\|_2}. \quad (4)$$

The properties of the Gram-Schmidt algorithm guarantee that $(\vec{z}_1, \vec{z}_2)$ is an orthonormal basis for $\operatorname{span}(\vec{x}, \vec{y})$.

4. Homework Process

With whom did you work on this homework? List the names and SIDs of your group members.

NOTE: If you didn't work with anyone, you can put "none" as your answer.