This homework is due at 11 PM on January 26, 2024.

Submission Format: Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned).

1. Least Squares

The Michaelis-Menten model for enzyme kinetics relates the rate $y$ of an enzymatic reaction to the concentration $x$ of a substrate, as follows:

$$y = \frac{\beta_1 x}{\beta_2 + x},$$

for constants $\beta_1, \beta_2 > 0$.

(a) Show that the model can be expressed as a linear relation between the values $1/y = y^{-1}$ and $1/x = x^{-1}$. Specifically, give an equation of the form $y^{-1} = w_1 + w_2 x^{-1}$, specifying the values of $w_1$ and $w_2$ in terms of $\beta_1$ and $\beta_2$.

(b) In general, reaction parameters $\beta_1$ and $\beta_2$ (and, thus, $w_1$ and $w_2$) are not known a priori and must be fit from data — for example, using least squares. Suppose you collect $m$ measurements $(x_i, y_i)$, $i = 1, \ldots, m$ over the course of a reaction. Formulate the least squares problem

$$\vec{w}^* = \underset{\vec{w}}{\text{argmin}} \| X \vec{w} - \vec{y} \|_2^2,$$

where $\vec{w}^* = \begin{bmatrix} w_1^* & w_2^* \end{bmatrix}^T$, and you must specify $X \in \mathbb{R}^{m \times 2}$ and $\vec{y} \in \mathbb{R}^m$. Specifically, your solution should include explicit expressions for $X$ and $\vec{y}$ as a function of $(x_i, y_i)$ values and a final expression for $\vec{w}^*$ in terms of $X$ and $\vec{y}$, which should contain only matrix multiplications, transposes, and inverses. Assume without loss of generality that $x_1 \neq x_2$.

(c) Assume that we have used the above procedure to calculate values for $w_1^*$ and $w_2^*$, and we now want to estimate $\tilde{\beta} = \begin{bmatrix} \tilde{\beta}_1 & \tilde{\beta}_2 \end{bmatrix}^T$. Write an expression for $\tilde{\beta}$ in terms of $w_1^*$ and $w_2^*$.

NOTE: This problem was taken (with some edits) from the textbook *Optimization Models* by Calafiore and El Ghaoui.
2. Subspaces and Dimensions

Consider the set $S$ of points $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that

$$x_1 + 2x_2 + 3x_3 = 0, \quad 3x_1 + 2x_2 + x_3 = 0. \quad (3)$$

(a) Find a $2 \times 3$ matrix $A$ for which $S$ is exactly the null space of $A$.

(b) Determine the dimension of $S$ and find a basis for it.
3. Vector Spaces and Rank

The rank of a \( m \times n \) matrix \( A \), \( \text{rank}(A) \), is the dimension of its range, also called span, and denoted \( \mathcal{R}(A) := \{ A\vec{x} : \vec{x} \in \mathbb{R}^n \} \).

(a) Assume that \( A \in \mathbb{R}^{m \times n} \) takes the form \( A = \vec{u}\vec{v}^\top \), with \( \vec{u} \in \mathbb{R}^m \), \( \vec{v} \in \mathbb{R}^n \), and \( \vec{u}, \vec{v} \neq \vec{0} \). (Note that a matrix of this form is known as a dyad.) Find the rank of \( A \).

HINT: Consider the quantity \( A\vec{x} \) for arbitrary \( \vec{x} \), i.e., what happens when you multiply any vector by matrix \( A \).

(b) Show that for arbitrary \( A, B \in \mathbb{R}^{m \times n} \),

\[
\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B),
\]

i.e., the rank of the sum of two matrices is less than or equal to the sum of their ranks.

HINT: First, show that \( \mathcal{R}(A + B) \subseteq \mathcal{R}(A) + \mathcal{R}(B) \), meaning that any vector in the range of \( A + B \) can be expressed as the sum of two vectors, each in the range of \( A \) and \( B \), respectively. Remember that for any matrix \( A \), \( \mathcal{R}(A) \) is a subspace, and for any two subspaces \( S_1 \) and \( S_2 \), \( \dim(S_1 + S_2) \leq \dim(S_1) + \dim(S_2) \). (Note that the sum of vector spaces \( S_1 + S_2 \) is not equivalent to \( S_1 \cup S_2 \), but is defined as \( S_1 + S_2 := \{ \vec{s}_1 + \vec{s}_2 : \vec{s}_1 \in S_1, \vec{s}_2 \in S_2 \} \).

(c) Consider an \( m \times n \) matrix \( A \) that takes the form \( A = UV^\top \), with \( U \in \mathbb{R}^{m \times k} \), \( V \in \mathbb{R}^{n \times k} \). Show that the rank of \( A \) is less or equal than \( k \).

HINT: Use parts (a) and (b), and remember that this decomposition can also be written as the dyadic expansion

\[
A = UV^\top = \begin{bmatrix} \vec{u}_1 & \ldots & \vec{u}_k \end{bmatrix} \begin{bmatrix} \vec{v}_1^\top \\ \vdots \\ \vec{v}_k^\top \end{bmatrix} = \sum_{i=1}^{k} \vec{u}_i \vec{v}_i^\top, \tag{5}
\]

for \( U = \begin{bmatrix} \vec{u}_1 & \ldots & \vec{u}_k \end{bmatrix} \) and \( V = \begin{bmatrix} \vec{v}_1 & \ldots & \vec{v}_k \end{bmatrix} \).

---

\(^1\text{This fact can be proved by taking a basis of } S_1 \text{ and extending it to a basis of } S_2 \text{ (during which we can only add at most } \dim(S_2) \text{ basis vectors). This extended basis must now also be a basis of } S_1 + S_2. \text{ Thus, } \dim(S_1 + S_2) \leq \dim(S_1) + \dim(S_2). \)
4. Norms

(a) Show that the following inequalities hold for any vector \( \vec{x} \in \mathbb{R}^n \):

\[
\frac{1}{\sqrt{n}} \|\vec{x}\|_2 \leq \|\vec{x}\|_{\infty} \leq \|\vec{x}\|_1 \leq \sqrt{n} \|\vec{x}\|_2 \leq n \|\vec{x}\|_{\infty} .
\] (6)

**NOTE:** We can interpret different norms as different ways of computing distance between two points \( \vec{x}, \vec{y} \in \mathbb{R}^2 \). The \( \ell^2 \) norm is the distance as the crow flies (i.e. point-to-point distance), the \( \ell^1 \) norm, also known as the Manhattan distance is the distance you would have to cover if you were to navigate from \( \vec{x} \) to \( \vec{y} \) via a rectangular street grid, and the \( \ell^\infty \) norm is the maximum distance that you have to travel in either the north-south or the east-west direction.

(b) We define the **sparsity** of the vector \( \vec{x} \) as the number of non-zero elements in \( \vec{x} \). This is also commonly known as the \( \ell^0 \) norm of the vector \( \vec{x} \), denoted by \( \|\vec{x}\|_0 \). Show that for any non-zero vector \( x \),

\[
\|\vec{x}\|_0 \geq \frac{\|\vec{x}\|_1^2}{\|\vec{x}\|_2^2} .
\] (7)

Find all vectors \( \vec{x} \) for which the lower bound is attained.
5. Homework Process

With whom did you work on this homework? List the names and SIDs of your group members.

NOTE: If you didn’t work with anyone, you can put “none” as your answer.