Homework 2

This homework is due at 11 PM on January 26, 2024.

Submission Format: Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned).

1. **Least Squares**

The Michaelis-Menten model for enzyme kinetics relates the rate y of an enzymatic reaction to the concentration x of a substrate, as follows:

$$y = \frac{\beta_1 x}{\beta_2 + x}, \tag{1}$$

for constants $\beta_1, \beta_2 > 0$.

(a) Show that the model can be expressed as a linear relation between the values $1/y = y^{-1}$ and $1/x = x^{-1}$. Specifically, give an equation of the form $y^{-1} = w_1 + w_2 x^{-1}$, specifying the values of w_1 and w_2 in terms of β_1 and β_2.

(b) In general, reaction parameters β_1 and β_2 (and, thus, w_1 and w_2) are not known a priori and must be fit from data — for example, using least squares. Suppose you collect m measurements (x_i, y_i), $i = 1, \ldots, m$ over the course of a reaction. Formulate the least squares problem

$$\vec{w}^\star = \underset{\vec{w}}{\operatorname{argmin}} \left\| X\vec{w} - \vec{y} \right\|_2^2, \tag{2}$$

where $\vec{w}^\star = \begin{bmatrix} w_1^\star & w_2^\star \end{bmatrix}^\top$, and you must specify $X \in \mathbb{R}^{m \times 2}$ and $\vec{y} \in \mathbb{R}^m$. Specifically, your solution should include explicit expressions for X and $\vec{y}$ as a function of (x_i, y_i) values and a final expression for $\vec{w}^\star$ in terms of X and $\vec{y}$, which should contain only matrix multiplications, transposes, and inverses.

Assume without loss of generality that $x_1 \neq x_2$.

(c) Assume that we have used the above procedure to calculate values for $w_1^\star$ and $w_2^\star$, and we now want to estimate $\widehat{\vec{\beta}} = \begin{bmatrix} \widehat{\beta}_1 & \widehat{\beta}_2 \end{bmatrix}^\top$. Write an expression for $\widehat{\vec{\beta}}$ in terms of $w_1^\star$ and $w_2^\star$.

NOTE: This problem was taken (with some edits) from the textbook *Optimization Models* by Calafiore and El Ghaoui.

2. Subspaces and Dimensions

Consider the set $\mathcal{S}$ of points $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that

$$x_1 + 2x_2 + 3x_3 = 0, \quad 3x_1 + 2x_2 + x_3 = 0. \tag{3}$$

(a) Find a 2×3 matrix A for which $\mathcal{S}$ is exactly the null space of A.

(b) Determine the dimension of $\mathcal{S}$ and find a basis for it.

3. Vector Spaces and Rank

The *rank* of a $m \times n$ matrix A, $\operatorname{rank}(A)$, is the dimension of its *range*, also called span, and denoted $\mathcal{R}(A) := \{A\vec{x} \,:\, \vec{x} \in \mathbb{R}^n\}$.

(a) Assume that $A \in \mathbb{R}^{m\times n}$ takes the form $A = \vec{u}\vec{v}^\top$, with $\vec{u} \in \mathbb{R}^m$, $\vec{v} \in \mathbb{R}^n$, and $\vec{u}, \vec{v} \neq \vec{0}$. (Note that a matrix of this form is known as a *dyad*.) Find the rank of A.

HINT: Consider the quantity $A\vec{x}$ for arbitrary $\vec{x}$, i.e., what happens when you multiply any vector by matrix A.

(b) Show that for arbitrary $A, B \in \mathbb{R}^{m\times n}$,

$$\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B), \tag{4}$$

i.e., the rank of the sum of two matrices is less than or equal to the sum of their ranks.

HINT: First, show that $\mathcal{R}(A + B) \subseteq \mathcal{R}(A) + \mathcal{R}(B)$, meaning that any vector in the range of $A + B$ can be expressed as the sum of two vectors, each in the range of A and B, respectively. Remember that for any matrix A, $\mathcal{R}(A)$ is a subspace, and for any two subspaces S_1 and S_2, $\dim(S_1 + S_2) \leq \dim(S_1) + \dim(S_2)$.[1] (Note that the sum of vector spaces $S_1 + S_2$ is not equivalent to $S_1 \cup S_2$, but is defined as $S_1 + S_2 := \{\vec{s_1} + \vec{s_2} | \vec{s_1} \in S_1, \vec{s_2} \in S_2\}$.)

(c) Consider an $m \times n$ matrix A that takes the form $A = UV^\top$, with $U \in \mathbb{R}^{m\times k}$, $V \in \mathbb{R}^{n\times k}$. Show that the rank of A is less or equal than k. *HINT: Use parts (a) and (b), and remember that this decomposition can also be written as the* dyadic expansion

$$A = UV^\top = \begin{bmatrix} \vec{u}_1 & \dots & \vec{u}_k \end{bmatrix} \begin{bmatrix} \vec{v}_1^\top \\ \vdots \\ \vec{v}_k^\top \end{bmatrix} = \sum_{i=1}^{k} \vec{u}_i \vec{v}_i^\top, \tag{5}$$

for $U = \begin{bmatrix} \vec{u}_1 & \dots & \vec{u}_k \end{bmatrix}$ and $V = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_k \end{bmatrix}$.

[1]This fact can be proved by taking a basis of S_1 and extending it to a basis of S_2 (during which we can only add *at most* $\dim(S_2)$ basis vectors). This extended basis must now also be a basis of $S_1 + S_2$. Thus, $\dim(S_1 + S_2) \leq \dim(S_1) + \dim(S_2)$.

4. Norms

(a) Show that the following inequalities hold for any vector $\vec{x} \in \mathbb{R}^n$:

$$\frac{1}{\sqrt{n}} \|\vec{x}\|_2 \leq \|\vec{x}\|_\infty \leq \|\vec{x}\|_2 \leq \|\vec{x}\|_1 \leq \sqrt{n} \|\vec{x}\|_2 \leq n \|\vec{x}\|_\infty . \tag{6}$$

NOTE: We can interpret different norms as different ways of computing distance between two points $\vec{x}, \vec{y} \in \mathbb{R}^2$. The ℓ^2 norm is the distance as the crow flies (i.e. point-to-point distance), the ℓ^1 norm, also known as the Manhattan distance is the distance you would have to cover if you were to navigate from $\vec{x}$ to $\vec{y}$ via a rectangular street grid, and the ℓ^∞ norm is the maximum distance that you have to travel in either the north-south or the east-west direction.

(b) We define the *sparsity* of the vector $\vec{x}$ as the number of non-zero elements in $\vec{x}$. This is also commonly known as the ℓ^0 norm of the vector $\vec{x}$, denoted by $\|\vec{x}\|_0$. Show that for any non-zero vector x,

$$\|\vec{x}\|_0 \geq \frac{\|\vec{x}\|_1^2}{\|\vec{x}\|_2^2}. \tag{7}$$

Find all vectors $\vec{x}$ for which the lower bound is attained.

5. Homework Process

With whom did you work on this homework? List the names and SIDs of your group members.

NOTE: If you didn't work with anyone, you can put "none" as your answer.

