This homework is due at 11 PM on 4 November 2022.
Submission Format: Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned).

1. Sensitivity and Dual Variables

In this problem, we look into the interpretation of dual variables as sensitivity parameters of the primal problem. Recall the canonical, convex primal problem

\[
\begin{align*}
\min_x & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad h_j(x) = 0, \quad j = 1, \ldots, p
\end{align*}
\]

where \(f_0, f_i\) are convex and \(h_j\) are affine (assume the problem has strong duality). Consider the perturbed problem

\[
\begin{align*}
\min_x & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq u_i, \quad i = 1, \ldots, m \\
& \quad h_j(x) = v_j, \quad j = 1, \ldots, p
\end{align*}
\]

and define

\[
p^\star(u, v) = \inf \{ f_0(x) \mid \exists x, f_i(x) \leq u_i \ \forall i, h_j(x) = v_j \ \forall j \}
\]

In words, \(p^\star(u, v)\) is the optimal value for the perturbed problem if it is feasible, and defined to be \(+\infty\) (infeasible) otherwise. Note \(p^\star(0, 0)\) is the original problem.

(a) Prove \(p^\star(u, v)\) is jointly convex in \((u, v)\).

HINT: Consider \(D = \{(x, u, v) \mid f_i(x) \leq u_i \ \forall i, h_j(x) = v_j \ \forall j\}\), which is the set of triples \((x, u, v)\) such that \(x\) is a feasible point for the perturbed problem with the perturbations \((u, v)\). Is \(D\) convex? Also, define \(F(x, u, v)\) to be a function that is equal to \(f_0(x)\) on \(D\) and \(+\infty\) otherwise.

(b) Assume that strong duality holds, and that the dual optimum is attained. Let \((\lambda^\star, \nu^\star)\) be the optimal dual variables for the dual of the unperturbed primal problem \((1)\). Then, show that for all \(u, v\), we have

\[
p^\star(u, v) \geq p^\star(0, 0) - u^\top \lambda^\star - v^\top \nu^\star.
\]

HINT: Let \(\hat{x}\) be a feasible point for the perturbed problem. Can you show that \(f_0(\hat{x}) \geq p^\star(0, 0) - \sum_{i=1}^m \lambda_i^\star u_i - \sum_{j=1}^p \nu_j^\star v_j\)?

(c) Suppose we only have 1 equality and 1 inequality constraint (that is, \(u, v\) are scalars). For each of the following situations, argue whether the value of \(p^\star(u, v)\) increases or decreases as compared to \(p^\star(0, 0)\) or whether we are unsure as to whether it increases or decreases.

i. If \(\lambda^\star\) is large and we pick \(u < 0\).

ii. If \(\lambda^\star\) is large and we pick \(u > 0\).

iii. If \(\nu^\star\) is large and positive (resp. negative) and we pick \(v < 0\) (resp. \(v > 0\)).
2. KKT with circles

Consider the problem

\[
\begin{align*}
\min_{\vec{x} \in \mathbb{R}^2} & \quad x_1^2 + x_2^2 \\
\text{s.t.} & \quad (x_1 - 1)^2 + (x_2 - 1)^2 \leq 2 \\
& \quad (x_1 - 1)^2 + (x_2 + 1)^2 \leq 2
\end{align*}
\]

where \( \vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top \in \mathbb{R}^2. \)

(a) Sketch the feasible region and the level sets of the objective function. Find the optimal point \( \vec{x}^* \) and the optimal value \( p^*. \)

(b) Does strong duality hold?

(c) Write the KKT conditions for this optimization problem. Do there exist Lagrange multipliers \( \lambda_1^* \) and \( \lambda_2^* \) that prove the optimality of \( \vec{x}^* \)?
3. Water Filling

Consider the following problem:

\[
\text{minimize} \quad - \sum_{i=1}^{n} \log(\alpha_i + x_i) \\
\text{subject to} \quad x \geq 0, \quad \mathbf{1}^\top x = 1,
\]

where \(\alpha_i > 0\) for each \(i = 1, \ldots, n\).

This problem arises in information theory, in allocating power to a set of \(n\) communication channels. The variable \(x_i\) represents the transmitter power allocated to the \(i\)th channel, and \(\log(\alpha_i + x_i)\) gives the capacity or communication rate of the channel, so the problem is to allocate a total power of one to the channels, in order to maximize the total communication rate.

(a) Verify that this is a convex optimization problem with differentiable objective and constraint functions. Find the domain \(D\) of the optimization problem.

(b) Let \(\lambda \in \mathbb{R}^n\) and \(\nu \in \mathbb{R}\) be the dual variables corresponding to the constraints \(x_i \geq 0, i = 1, \ldots, n\) and \(\mathbf{1}^\top x = 1\), respectively. Write a Lagrangian for the optimization problem based on these dual variables.

(c) Write the KKT conditions for the problem.

(d) Since our problem is a convex optimization problem with differential objective and constraint functions, the KKT conditions provide sufficient conditions for optimality. Hence, we know that if we can find \(x^*\) and \((\lambda^*, \nu^*)\) that verify the KKT conditions, then \(x^*\) will be a primal optimal point, \((\lambda^*, \nu^*)\) will be dual optimal. We therefore attempt to find solutions for the KKT conditions. As a first step, show how to simplify the KKT conditions so that they are expressed in terms of only \(x^*\) and \(\nu^*\), i.e. we show how \(\lambda^*\) can be eliminated from these conditions.

(e) Solve for \(x_i^*, 1 \leq i \leq n\), in terms of \(\nu^*\) from the simplified KKT conditions derived in the preceding part of this question.

(f) Show that there is a unique dual optimizer \(\nu^*\), and describe an algorithm for finding it.
4. Does strong duality hold?

Consider

\[ \min_{(x,y) \in D} e^{-x} \]  
\[ \text{s.t. } x^2/y \leq 0 \]

where \( D = \{(x,y) \mid y > 0\} \).

(a) Prove the problem is convex. Find the optimal value. \textit{HINT: To prove the constraint function is convex, you will have to prove it is convex with respect to the vector \([x \ y]^\top\). Consider computing the Hessian of the constraint function, its determinant and trace, and show that it is PSD by analyzing signs of its eigenvalues.}

(b) Next, we will proceed to find an optimal solution and an optimal value for the dual problem. The Lagrangian dual function \( g(\lambda) \), can be written as:

\[ g(\lambda) = \inf_{(x,y) \in D} \left( e^{-x} + \lambda \frac{x^2}{y} \right). \]

Explain why \( g(\lambda) \) is lower bounded by 0 for \( \lambda \geq 0 \). \textit{Note: Here we are not dualizing the constraint \( y > 0 \) that is in the definition of \( D \) — this is only dualizing the other constraint.}

(c) Show that \( g(\lambda) = 0 \) for \( \lambda \geq 0 \). \textit{HINT: To show that the infimum in Equation (16) is 0, we want to show there exist \((x,y)\) such that both \( e^{-x} \) and \( \lambda \frac{x^2}{y} \) can get arbitrarily close to 0. HINT: Consider a sequence \( \{x_k\} \) going to \(+\infty\) and a sequence \( \{y_k\} \) also going to \(+\infty\) such that \( \lim_{k \to \infty} \frac{x_k^2}{y_k} = 0 \). Simply put, we want to drive \( x \) to infinity in order to drive \( e^{-x} \) to 0, while having \( y \) grow faster than \( x^2 \), so that the second term also goes to 0.}

(d) Now, write the dual problem and find an optimal solution \( \lambda^* \) and an optimal value \( d^* \) for the dual problem using the results above. What is the duality gap?

(e) Does Slater’s Condition hold for this problem? Does Strong Duality hold?
5. Dual of the dual of a linear program

Consider a standard linear program $P$:

$$\begin{align*}
\min_{\vec{x}} & \quad \vec{c}^T \vec{x} \\
\text{s.t.} & \quad A\vec{x} = \vec{b} \\
& \quad \vec{x} \geq 0.
\end{align*}$$

where $\vec{x}, \vec{c} \in \mathbb{R}^n, \vec{b} \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$.

(a) Formulate the Lagrangian of the problem $P$, and write the dual problem.

Note: The dual problem should not have the variable $\vec{x}$.

(b) Express the dual problem as an equivalent minimization problem. Find the dual of this minimization problem, i.e., the dual of the dual. Compare it to the original linear program formulation.
6. Homework Process

With whom did you work on this homework? List the names and SIDs of your group members.

*NOTE:* If you didn’t work with anyone, you can put “none” as your answer.