EECS 127/227AT | Optimization Models in Engineering | UC Berkeley | Spring 2024

Homework 10

Self grades are due at 11 PM on April 12, 2024.

1. Does Strong Duality Hold?

Consider

$$\min_{(x,y)\in\mathcal{D}} \quad e^{-x} \tag{1}$$

$$\text{s.t.} \quad x^2/y \le 0 \tag{2}$$

where $\mathcal{D} = \{(x,y) \mid y > 0\}$.

(a) Prove the problem is convex. Find the optimal value. *HINT: To prove the constraint function is convex, you will have to prove it is convex with respect to the vector* $\begin{bmatrix} x & y \end{bmatrix}^\top$. *Consider computing the Hessian of the constraint function, its determinant and trace, and show that it is PSD by analyzing signs of its eigenvalues.*

Solution: The second derivative of the objective function is e^{-x} which is non-negative thus the objective is a convex function.

Furthermore, the constraint is jointly convex as can be verified by showing the Hessian is PSD (additionally one may notice this is the perspective function of x^2 which will be convex).

The Hessian for $g(x,y) = \frac{x^2}{y}$ is given by,

$$\nabla^2 g(x,y) = \begin{bmatrix} \frac{2}{y} & -\frac{2x}{y^2} \\ -\frac{2x}{y^2} & \frac{2x^2}{y^3} \end{bmatrix}. \tag{3}$$

Suppose λ_1, λ_2 are the eigenvalues of $\nabla^2 g(x,y)$. The determinant of the Hessian is 0 which gives us $\lambda_1\lambda_2 = 0$. Further trace of the Hessian is $\frac{2}{y} + \frac{2x^2}{y^3} > 0$ (since $y > 0$) which gives $\lambda_1 + \lambda_2 > 0$. Thus one eigenvalue must be positive and the other must be 0 which shows that the Hessian is positive semidefinite. Hence the problem is convex. Furthermore, the only feasible value of x is $x = 0$. Hence the optimal value is $e^{-0} = 1$.

(b) Next, we will proceed to find an optimal solution and an optimal value for the dual problem. The Lagrangian dual function $g(\lambda)$, can be written as:

$$g(\lambda) = \inf_{(x,y)\in\mathcal{D}} e^{-x} + \lambda\frac{x^2}{y}. \tag{4}$$

Explain why $g(\lambda)$ is lower bounded by 0 for $\lambda \ge 0$. *NOTE*: Here we are not dualizing the constraint $y > 0$ that is in the definition of $\mathcal{D}$ — this is only dualizing the other constraint.

Solution: This is true since both terms in the sum are non-negative because $y > 0$.

(c) Show that $g(\lambda) = 0$ for $\lambda \ge 0$. *HINT: To show that the infimum in (4) is 0, we want to show there exist* (x,y) *such that both* e^{-x} *and* $\lambda\frac{x^2}{y}$ *can get arbitrarily close to 0. HINT: Consider a sequence* $\{x_k\}$ *going to* $+\infty$ *and a sequence* $\{y_k\}$ *also going to* $+\infty$ *such that* $\lim_{k\to\infty}\frac{x_k^2}{y_k} = 0$. *Simply put, we want to drive* x *to infinity in order to drive* e^{-x} *to* 0, *while having* y *grow faster than* x^2, *so that the second term also goes to* 0.

Solution: To show that the infimum is 0, we pick any sequence $\{x_k\}$ going to $+\infty$ and pick a sequence $\{y_k\}$ also going to $+\infty$ such that $\lim_{k\to\infty} \frac{x_k^2}{y_k} = 0$. One example of such sequence pair is $y_k = x_k^4$ and $x_k = 2k$.

This gives $\lim_{k\to\infty} \mathrm{e}^{-x} + \frac{x_k^2}{y_k} = 0$, so $g(\lambda) = \inf_{(x,y)\in\mathcal{D}} \mathrm{e}^{-x} + \lambda\frac{x^2}{y} = 0$. Note that the (x_k, y_k) pair as $k \to \infty$ does not satisfy the constraint of the primal problem $x^2/y \leq 0$ but allows x^2/y to get arbitrarily close to 0.

(d) Now, write the dual problem and find an optimal solution λ^* and an optimal value d^* for the dual problem using the results above. What is the duality gap?

Solution: The Lagrange dual problem is

$$d^* = \sup_{\lambda\geq 0} g(\lambda) \tag{5}$$

where $g(\lambda) = \inf_{(x,y)\in\mathcal{D}} \; \mathrm{e}^{-x} + \lambda x^2/y$. Note $g(\lambda) = 0$ from previous part. It follows from the previous problems that $d^* = 0$ and any $\lambda \geq 0$ is optimal. The duality gap is 1.

(e) Does Slater's Condition hold for this problem? Does Strong Duality hold?

Solution:

Method 1: While the primal problem is convex, we cannot find a point that is strictly in the interior of the domain and satisfies the constraint as needed for Slater's condition.

Specifically, for Slater's condition to hold we need the existence of an (x, y) pair such that $x^2/y < 0$. Note there is no such pair (x, y) since $y > 0$ and $x^2 \geq 0$. Hence Slater's condition does not hold for this problem.

Method 2: From the previous parts we saw that $p^* \neq d^*$, and thus strong duality does not hold. Furthermore, the problem is convex.

For convex problems we know that if Slater's condition holds then we must have strong duality (i.e Slater's is a sufficient condition). However since strong duality does not hold it implies that Slater's does not hold.

Note that this problem is an example illustrating that **convexity alone is not enough to guarantee strong duality for an optimization problem**.

2. About General Optimization

In this exercise, we test your understanding of the general framework of optimization and its language. We consider an optimization problem in standard form:

$$p^\star = \min_{\vec{x} \in \mathbb{R}^n} f_0(\vec{x}) \ : \ f_i(\vec{x}) \le 0, \ \ i = 1, \ldots, m. \tag{6}$$

In the following we denote by $\mathcal{X}$ the feasible set. Note that the feasible set is a subset of $\mathbb{R}^n$ that satisfies the inequalities $f_i(\vec{x}) \le 0$, i.e $\mathcal{X} = \{\vec{x} \in \mathbb{R}^n \mid f_i(\vec{x}) \le 0, i = 1, \ldots, m\}$. We make no assumption about the convexity of $f_0(\vec{x})$ and $f_i(\vec{x}), \ i = 1, \ldots, m$. For the following statements, answer whether the statement is true or false and provide a proof or counter-example.

(a) A general optimization problem can be expressed as one with a linear objective.

Solution: The statement is true:

$$p^\star = \min_{\vec{x} \in \mathcal{X}, t} t \ : \ t \ge f_0(\vec{x}). \tag{7}$$

(b) A general optimization problem can be expressed as an unconstrained problem with a different objective function which could possibly take a value of ∞ for some values of $\vec{x}$.

Solution: Again the statement is true: let us define

$$g(\vec{x}) := \begin{cases} f_0(\vec{x}) & \text{if } \vec{x} \in \mathcal{X}, \\ +\infty & \text{otherwise} \end{cases} \tag{8}$$

Then

$$p^\star = \min_{\vec{x}} g(\vec{x}). \tag{9}$$

(c) A general optimization problem can be recast as minimizing a linear objective subject to (possibly infinitely many) linear constraints.

Solution: This is true again; we have

$$p^\star = \max_{t} t \ : \ t \le f_0(\vec{x}) \text{ for every } \vec{x} \in \mathcal{X}. \tag{10}$$

Alternatively,

$$p^\star = -\min_{t} -t \ : \ t \le f_0(\vec{x}) \text{ for every } \vec{x} \in \mathcal{X}. \tag{11}$$

It may not be convincing that the above constraints are linear. To see why, we consider the restricted case that $\mathcal{X} = \{1, 2, 3\}$; then our reformulation becomes

$$\begin{aligned} p^\star = \max_{t} \quad & t && (12) \\ \text{s.t.} \quad & t \le f_0(1) && (13) \\ & t \le f_0(2) && (14) \\ & t \le f_0(3) && (15) \end{aligned}$$

which are just 3 separate linear constraints. Adding more linear constraints in this way (even uncountably many of them) doesn't change the fact that each constraint is linear.

(d) If any of the constraint inequalities is strict (and therefore not active) at the optimum point, then we can remove the constraint from the original problem and obtain the same optimum value.

Note: Review the definition of active constraints from the textbooks: Boyd Section 4.1.1 and El Ghaoui Section 8.3.

Hint: Consider the problem

$$\min_x \; f(x) \; = \begin{cases} x^2 & \text{if } |x| \le 1, \\ -1 & \text{otherwise} \end{cases} \tag{16}$$

$$\text{such that} \;\; |x| \le 1 \tag{17}$$

Solution: This is *not* true in general. Consider the problem

$$p^\star := \min_x \; f_0(x) \; : \; |x| \le 1, \tag{18}$$

where

$$f_0(x) = \begin{cases} x^2 & \text{if } |x| \le 1, \\ -1 & \text{otherwise.} \end{cases} \tag{19}$$

The constraint $|x| \le 1$ is not active at the optimum $x^\star = 0$, and $p^\star = 0$. However, if we remove it, the new optimal value becomes -1.

(e) Now, suppose for the formulation in (6), $f_0(\vec{x})$ is a convex function, $\mathcal{X}$ is a convex set, and all $f_i(\vec{x})$ are convex and continuous functions. Suppose $p^\star$ is achieved at a point $\vec{x}^\star$ where for some i, $f_i(\vec{x}^\star) < 0$. Prove that we can remove this inequality constraint and still retain the same optimum. In other words, show that

$$p^\star = \min_{\vec{x} \in \mathbb{R}^n} \; f_0(\vec{x}) \; : \; f_j(\vec{x}) \le 0, \;\; j = 1, \ldots i-1, i+1, \ldots, m. \tag{20}$$

HINT: Argue by contradiction that if by removing the inequality constraint $f_i(\vec{x}) \le 0$, we achieve a different optimal for the problem in (6) at some point $\vec{\bar{x}}$ that satisfies $f_i(\vec{\bar{x}}) > 0$, then there exists a point $\vec{y}$ between $\vec{\bar{x}}$ and $\vec{x}^\star$ that is feasible to the original problem in (6). Use the continuity of f_i and the intermediate value theorem to come up with a $\vec{y}$ then show that it must be more optimal than $\vec{x}^\star$ in (6).

Solution: Suppose for contradiction that by removing the inequality constraint $f_i(\vec{x}) \le 0$, we achieve a more optimal result (we can not achieve a less optimal result by relaxing the feasible set). Concretely, let

$$\vec{\bar{x}} \in \operatorname*{argmin}_{\vec{x} \in \mathbb{R}^n} \; f_0(\vec{x}) \; : \; f_j(\vec{x}) \le 0, \;\; j = 1, \ldots i-1, i+1, \ldots, m. \tag{21}$$

such that

$$\vec{\bar{x}} \notin \operatorname*{argmin}_{\vec{x} \in \mathbb{R}^n} \; f_0(\vec{x}) \; : \; f_i(\vec{x}) \le 0, \;\; i = 1, \ldots, m \tag{22}$$

Then, we must have that $f_i(\vec{\bar{x}}) > 0$. Since $f_i(\vec{x}^\star) < 0$, by the continuity of f_i, there must exist some $\theta \in [0, 1]$ such that $f_i(\theta\vec{x}^\star + (1-\theta)\vec{\bar{x}}) = 0$.

By the convexity of f_0, we have that

$$\begin{aligned} f_0(\theta\vec{x}^\star + (1-\theta)\vec{\bar{x}}) &\le \theta f_0(\vec{x}^\star) + (1-\theta) f_0(\vec{\bar{x}}) && (23) \\ &< f_0(\vec{x}^\star) && (24) \end{aligned}$$

This contradicts our assumption that $p^\star$ is achieved at $\vec{x}^\star$ since $\theta\vec{x}^\star + (1-\theta)\vec{\bar{x}}$ is more optimal.

3. Formulating Optimization problems

(a) **Linear Separability.** Let $(\vec{x}_i, y_i)$ be given data points with $\vec{x}_i \in \mathbb{R}^n$ and binary labels $y_i \in \{-1, 1\}$. We want to know if it is possible to find a hyperplane $\mathcal{L} = \{\vec{x} \in \mathbb{R}^n : \vec{h}^\top \vec{x} + b = 0\}$ that separates all the points with labels $y_i = -1$ from all the points with labels $y_i = 1$. In other words, can we find a vector $\vec{h} \in \mathbb{R}^n$ and a scalar $b \in \mathbb{R}$ such that $\vec{h}^\top \vec{x}_i + b \leq 0$ for all i such that $y_i = 1$ and $\vec{h}^\top \vec{x}_i + b > 0$ for all i such that $y_i = -1$. We want to cast this task as the following LP

$$p^\star = \min_{\vec{h}, b, z} \quad f_0(\vec{h}, b, z) \tag{25}$$

$$s.t. \quad \vec{h}^\top \vec{x}_i + b \leq 0 \qquad \forall i : y_i = 1 \tag{26}$$

$$\vec{h}^\top \vec{x}_i + b \geq z \qquad \forall i : y_i = -1 \tag{27}$$

Complete this formulation by specifying a linear objective function f_0. What does the solution $p^\star$ say about the existence of the separating hyperplane?

Solution: With the choice of the objective function

$$f_0(\vec{h}, b, z) = -z, \tag{28}$$

the separating hyperplane exists if $p^\star < 0$.

To see this note that $p^\star < 0$ if and only if the optimal solution to the problem $(\vec{h}^\star, b^\star, z^\star)$ is such that $z^\star > 0$. Now consider the hyperplane $\mathcal{L}^\star = \{\vec{x} \in \mathbb{R}^n : \vec{h}^{\star\top} \vec{x} + b^\star = 0\}$. From feasibility of the optimal solution, we can see that

$$\vec{h}^{\star\top} \vec{x}_i + b^\star \leq 0 \qquad \forall i : y_i = 1 \tag{29}$$

$$\vec{h}^{\star\top} \vec{x}_i + b^\star \geq z^\star > 0 \qquad \forall i : y_i = -1. \tag{30}$$

which is the definition of linear separability. Thus $\mathcal{L}^\star$ is indeed a separating hyperplane for this data.

(b) **Chebyshev Center.** Let $\mathcal{P} \subset \mathbb{R}^n$ be a non-empty polyhedron defined as the intersection of m hyperplanes $\mathcal{P} = \{\vec{x} : \vec{a}_i^\top \vec{x} \leq b_i \; \forall i = 1, 2, \ldots, m\}$. We define the Euclidean ball in $\mathbb{R}^n$ with radius R and center $\vec{x}_0$ as the set $\mathcal{B}(\vec{x}_0, R) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{x}_0\|_2 \leq R\}$. We want to find a point $\vec{x}_0 \in \mathcal{P}$ that is the center of the largest Euclidean ball contained in $\mathcal{P}$. Cast this problem as an LP.

Solution: Any point $\vec{x} \in \mathcal{B}(\vec{x}_0, R)$ can we expressed as $\vec{x} = \vec{x}_0 + \vec{u}$ where $\|\vec{u}\|_2 \leq R$. To satisfy the condition that $\mathcal{B}(\vec{x}_0, R) \subset \mathcal{P}$ we need for all $\vec{u} \in \mathbb{R}^n$ with norm $\|\vec{u}\|_2 \leq R$:

$$\vec{a}_i^\top (\vec{x}_0 + \vec{u}) \leq b_i \qquad \forall i = 1, 2, \ldots, m \tag{31}$$

We take the maximum over $\vec{u}$ of both sides to get the equivalent condition

$$\max_{\|\vec{u}\|_2 \leq R} \left(\vec{a}_i^\top (\vec{x}_0 + \vec{u})\right) \leq b_i \tag{32}$$

$$\vec{a}_i^\top \vec{x}_0 + \max_{\|\vec{u}\|_2 \leq R} \left(\vec{a}_i^\top \vec{u}\right) \leq b_i \qquad \forall i = 1, 2, \ldots, m \tag{33}$$

The inner product $\vec{a}_i^\top \vec{u}$ is maximized when $\vec{u}$ is the longest possible vector along the direction of $\vec{a}_i$, thus

$$\max_{\|\vec{u}\|_2 \leq R} \left(\vec{a}_i^\top \vec{u}\right) = \vec{a}_i^\top \left(\frac{R}{\|\vec{a}_i\|_2} \vec{a}_i\right) \tag{34}$$

$$= R \|\vec{a}_i\|_2 \tag{35}$$

This gives the following conditions

$$\vec{a}_i^\top \vec{x}_0 + R \|\vec{a}_i\|_2 \leq b_i \qquad \forall i = 1, 2, \ldots, m \tag{36}$$

Now we can write the problem of finding the largest ball enclosed in $\mathcal{P}$ as

$$\min_{\vec{x}_0, R} - R \tag{37}$$

$$s.t. \quad \vec{a}_i^\top \vec{x}_0 + R \|\vec{a}_i\|_2 \leq b_i \qquad \forall i = 1, 2, \ldots, m \tag{38}$$

4. Water Filling

Consider the following problem:

$$\begin{aligned} \text{minimize} \quad & -\sum_{i=1}^{n} \log(\alpha_i + x_i) && (39) \\ \text{subject to} \quad & \vec{x} \geq 0,\ \vec{1}^\top \vec{x} = c, && (40) \end{aligned}$$

where $\alpha_i > 0$ for each $i = 1, \ldots, n$.

This problem arises in information theory, in allocating power to a set of n communication channels. The variable x_i represents the transmitter power allocated to the ith channel, and $\log(\alpha_i + x_i)$ gives the capacity or communication rate of the channel, so the problem is to allocate a total power of c to the channels, in order to maximize the total communication rate.

(a) Verify that this is a convex optimization problem with differentiable objective and constraint functions. Find the domain $\mathcal{D}$ of the objective function $-\sum_{i=1}^{n} \log(\alpha_i + x_i)$ where it is well defined.

Solution: The objective function is convex with domain $\{\vec{x} : x_i + \alpha_i > 0 \forall i\}$, which is an open set in $\mathbb{R}^n$, and the objective function is differentiable on its domain. The n inequalities are determined by the function $-x_i$, $1 \leq i \leq n$, each of which is convex with domain $\mathbb{R}^n$ (note that these have to be thought of as function on $\mathbb{R}^n$) and is differentiable on its domain. There is one equality constraint, determined by the function $\vec{1}^\top \vec{x} = c$, which is convex with domain $\mathbb{R}^n$ and is differentiable on its domain.

The domain $\mathcal{D}$ of the optimization problem is the intersection of the domains of the objective and the constraint functions. Hence we have

$$\mathcal{D} = \{\vec{x} : x_i + \alpha_i > 0 \forall i\} \tag{41}$$

(b) Let $\vec{\lambda} \in \mathbb{R}^n$ and $\nu \in \mathbb{R}$ be the dual variables corresponding to the constraints $x_i \geq 0, i = 1, \ldots, n$ and $\vec{1}^\top \vec{x} = c$, respectively. Write a Lagrangian for the optimization problem based on these dual variables.

Solution: We can write the Lagrangian as:

$$\mathcal{L}(\vec{x}, \vec{\lambda}, \nu) = -\sum_{i=1}^{n} \log(\alpha_i + x_i) - \sum_{i=1}^{n} \lambda_i x_i + \nu(\vec{1}^\top \vec{x} - c). \tag{42}$$

The domain of the Lagrangian is $\mathcal{D} \times \mathbb{R}^n \times \mathbb{R}$.

(c) Write the KKT conditions for the problem.

Solution: The KKT conditions are the following:

i. Stationarity:

$$-\frac{1}{\alpha_i + x_i^*} - \lambda_i^* + \nu^* = 0, \quad i = 1, \ldots, n \tag{43}$$

ii. Primal feasibility:

$$\vec{x}^* \geq 0, \quad i = 1, \ldots, n, \quad \vec{1}^\top \vec{x}^* = c \tag{44}$$

iii. Dual feasibility:

$$\lambda^* \geq 0, \quad i = 1, \ldots, n \tag{45}$$

iv. Complementary slackness:

$$\lambda_i^* x_i^* = 0, \quad i = 1, \ldots, n \tag{46}$$

(d) Since our problem is a convex optimization problem with differentiable objective and constraint functions, the KKT conditions provide sufficient conditions for optimality. Hence, we know that if we can find $\vec{x}^*$ and $(\vec{\lambda}^*, \nu^*)$ that satisfy the KKT conditions, then $\vec{x}^*$ will be a primal optimal point, $(\vec{\lambda}^*, \nu^*)$ will be dual optimal. We therefore attempt to find solutions for the KKT conditions. As a first step, show how to simplify the KKT conditions so that they are expressed in terms of only $\vec{x}^*$ and ν^*, i.e. show how $\vec{\lambda}^*$ can be eliminated from these conditions.

Solution: The Lagrangian stationary conditions tell us how to write λ_i^* in terms of ν^* and x_i^*. However, the corresponding dual feasibility conditions, which require $\lambda_i^* \geq 0$, will manifest themselves as a new constraint on the pair (ν^*, x_i^*) when we do this. The corresponding complementary slackness condition will also now need to be expressed in terms of the pair (ν^*, x_i^*).
Eliminating λ^* in this way leads to the following simplified set of KKT conditions.

$$\vec{1}^\top \vec{x}^* = c \tag{47}$$

$$x_i^*(\nu^* - \frac{1}{\alpha_i + x_i^*}) = 0, \quad 1 \leq i \leq n, \tag{48}$$

$$x_i^* \geq 0, \quad 1 \leq i \leq n, \tag{49}$$

$$\nu^* \geq \frac{1}{\alpha_i + x_i^*}, \quad 1 \leq i \leq n. \tag{50}$$

(e) Solve for $x_i^*, 1 \leq i \leq n$, in terms of ν^* from the simplified KKT conditions derived in the previous subpart.

Solution: We can find the solution by considering two cases:

- If $\nu^* < 1/\alpha_i$, then we must have $x_i > 0$. as can be seen from the condition $\nu^* \geq 1/(\alpha_i + x_i^*)$. Then complementary slackness gives $\nu^* = 1/(\alpha_i + x_i^*)$, or equivalently, $x_i^* = -\alpha_i + 1/\nu^*$.
- If $\nu^* > 1/\alpha_i$, by complementary slackness, we must have $x_i^* = 0$. To see this, assume $x_i^* > 0$:

$$x_i^* > 0 \implies \nu^* = \frac{1}{\alpha_i + x_i^*} \implies \nu^* \leq \frac{1}{\alpha_i}, \tag{51}$$

  which leads to a contradiction.

As a result, we can write the optimal solution $\vec{x}^*$ in terms of ν^* as:

$$x_i^* = \begin{cases} -\alpha_i + 1/\nu^* & \text{if } \nu^* < 1/\alpha_i, \\ 0 & \text{if } \nu^* \geq 1/\alpha_i, \end{cases} \tag{52}$$

$$= \max\{0, -\alpha_i + 1/\nu^*\} \tag{53}$$

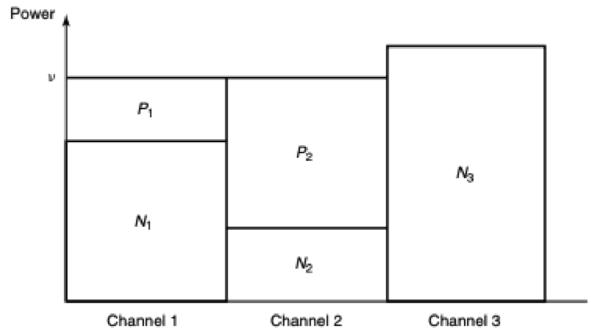


Figure 1: This graphic depicts a solution to the water-filling problem. On the x-axis we have n communication channels and on the y-axis we have the power in each channel. There is a base amount of noise N_i, which for us corresponds to α_i. Water-filling tells us that we should fill each channel until $\frac{1}{\nu^*}$, adding $\frac{1}{\nu^*} - \alpha_i$ power (in this graphic written as P_i), unless α_i already exceeds $\frac{1}{\nu^*}$. One algorithm for achieving this is to allot power to the channel with the least noise until it matches the channel with the second-least noise. Then we fill both simultaneously until they match the level of the channel with the third-least noise. Repeating this process until we run out of power to allot. This distribution of power is akin to filling connected basins with water, hence the name 'water filling'. Figure taken from *Elements of Information Theory* by Cover and Thomas.

5. Linear Programs

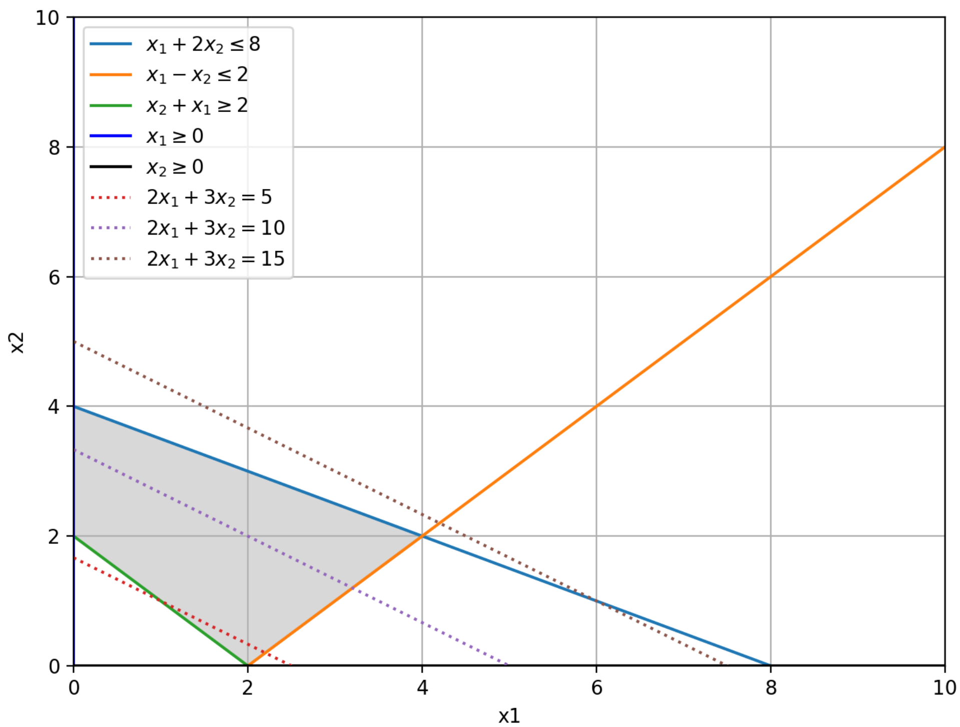
Consider the following linear program:

$$\max_{x_1, x_2 \in \mathbb{R}} \quad 2x_1 + 3x_2 \tag{54}$$
$$\text{s.t.} \quad x_1 + 2x_2 \leq 8 \tag{55}$$
$$x_1 - x_2 \leq 2 \tag{56}$$
$$x_2 + x_1 \geq 2 \tag{57}$$
$$x_1 \geq 0 \tag{58}$$
$$x_2 \geq 0 \tag{59}$$

(a) Sketch the feasible region of the linear program as well as the 5-, 10-, and 15-level sets of the objective function.

HINT: Recall that for $f : \mathbb{R}^n \mapsto \mathbb{R}$ and $\alpha \in \mathbb{R}$, the α-level set of f is defined as $\{\vec{x} \in \mathbb{R}^n : f(\vec{x}) = \alpha\}$.

Solution:



The shaded area in the figure is the feasible set.

(b) Express the linear program in the following form:

$$\max_{\vec{x} \in \mathbb{R}^n} \quad \vec{c}^\top \vec{x} \tag{60}$$
$$\text{s.t.} \quad A\vec{x} \leq \vec{b} \tag{61}$$

Specify the values of $\vec{x}$, $\vec{c}$, A, and $\vec{b}$.

Solution:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{62}$$

$$\vec{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \tag{63}$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ -1 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{64}$$

$$\vec{b} = \begin{bmatrix} 8 \\ 2 \\ -2 \\ 0 \\ 0 \end{bmatrix} \tag{65}$$

(c) Express the linear program in the following form:

$$\min_{\vec{y} \in \mathbb{R}^n} \quad \vec{c}^\top \vec{y} \tag{66}$$

$$\text{s.t.} \quad A\vec{y} = \vec{b} \tag{67}$$

$$\vec{y} \geq 0 \tag{68}$$

Specify the values of $\vec{y}$, $\vec{c}$, A, and $\vec{b}$.

HINT: Consider adding additional slack variables to the optimization problem.

Solution: Rewrite the linear program into the following by introducing slack variables s_1, s_2, s_3.

$$\max_{x_1, x_2, s_1, s_2, s_3 \in \mathbb{R}} \quad 2x_1 + 3x_2 \tag{69}$$

$$\text{s.t.} \quad x_1 + 2x_2 + s_1 = 8 \tag{70}$$

$$x_1 - x_2 + s_2 = 2 \tag{71}$$

$$x_2 + x_1 - s_3 = 2 \tag{72}$$

$$x_1 \geq 0 \tag{73}$$

$$x_2 \geq 0 \tag{74}$$

$$s_1 \geq 0 \tag{75}$$

$$s_2 \geq 0 \tag{76}$$

$$s_3 \geq 0 \tag{77}$$

(78)

Converting $\max_{x_1,x_2,s_1,s_2,s_3\in\mathbb{R}} 2x_1 + 3x_2$ to $\min_{x_1,x_2,s_1,s_2,s_3\in\mathbb{R}} -2x_1 - 3x_2$. We have

$$\vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} \tag{79}$$

$$\vec{c} = \begin{bmatrix} -2 \\ -3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{80}$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 \end{bmatrix} \tag{81}$$

$$\vec{b} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix} \tag{82}$$

(d) List the extreme points or vertices of the feasible region of the linear program given by equations (69)–(78).

Solution: By observation of the feasible set plotted, the extreme points are $(0,2)$, $(0,4)$, $(2,0)$, $(4,2)$.

(e) Find the optimal value $p^\star$ and the optimal point $\vec{x}^\star = (x_1^\star, x_2^\star)$ of the linear program given by equations (69)–(78).

HINT: Recall that for a linear program with a bounded feasible region, at least one optimal point is a vertex of the feasible region.

Solution: By observation of the feasible set and the level sets of the objective function, $\vec{x}^* = (4,2)$ and $p^* = 14$.

6. Homework Process

With whom did you work on this homework? List the names and SIDs of your group members.

NOTE: If you didn't work with anyone, you can put "none" as your answer.

