1. Fun with Hyperplanes

In this problem we work with hyperplanes, which are key components of linear programming as well as future topics such as support vector machines.

(a) Sketch the hyperplane $H \doteq \{ \vec{x} \in \mathbb{R}^2 \mid \begin{bmatrix} 1 & 1 \end{bmatrix} \vec{x} = 2 \}$.

(b) Let $\vec{c} \in \mathbb{R}^n$ be nonzero, and let $H \doteq \{ \vec{x} \in \mathbb{R}^n \mid \vec{c}^T \vec{x} = 0 \}$. Show that $H$ is a linear subspace of $\mathbb{R}^n$. What is $\dim(H)$?

(c) Let $\vec{c} \in \mathbb{R}^n$ be nonzero, and let $H \doteq \{ \vec{x} \in \mathbb{R}^n \mid \vec{c}^T \vec{x} = 0 \}$. Suppose $\vec{x}_* \in \mathbb{R}^n$ is on one side of the hyperplane, i.e., $\vec{c}^T \vec{x}_* > 0$. Give any vector which is on the other side of the hyperplane but not on the hyperplane itself.

(d) Let $\vec{c} \in \mathbb{R}^n$ be nonzero, and let $\vec{x}_0 \in \mathbb{R}^n$ be arbitrary. Let $H \doteq \{ \vec{x} \in \mathbb{R}^n \mid \vec{c}^T (\vec{x} - \vec{x}_0) = 0 \}$. Suppose $\vec{x}_* \in \mathbb{R}^n$ is on one side of the hyperplane. Give any vector which is on the other side of the hyperplane but not on the hyperplane itself.

(e) Let $\vec{x}_0 \in \mathbb{R}^n$ be arbitrary. For a vector $\vec{c} \in \mathbb{R}^n$, let $H(\vec{c}) \doteq \{ \vec{x} \in \mathbb{R}^n \mid \vec{c}^T (\vec{x} - \vec{x}_0) = 0 \}$. Show that $\vec{0} \in H(\vec{c})$ for every $\vec{c} \in \mathbb{R}^n$ if and only if $\vec{x}_0 = \vec{0}$. 

2. LP at Boundary

Consider the LP:

\[
\begin{align*}
\min_{\vec{x} \in \mathbb{R}^n} & \quad \vec{c}^\top \vec{x} \\
\text{s.t.} & \quad A\vec{x} \leq \vec{b}.
\end{align*}
\]

Here we have non-zero \(\vec{c} \in \mathbb{R}^n\), \(\vec{x} \in \mathbb{R}^n\), \(A \in \mathbb{R}^{m \times n}\), and \(\vec{b} \in \mathbb{R}^m\). The feasible set forms a polyhedron

\[
\mathcal{P} = \{\vec{x} \in \mathbb{R}^n \mid \vec{a}_i^\top \vec{x} - b_i \leq 0, \ 1 \leq i \leq n\}
\]

\[
= \bigcap_{i=1}^{m} \{\vec{x} \in \mathbb{R}^n \mid \vec{a}_i^\top \vec{x} - b_i \leq 0\}.
\]

In the second line, \(\mathcal{P}\) is defined as the intersection of half-spaces. When this set is bounded, it is often referred to as a polytope instead.

In this problem we will prove facts about linear programs, including the crucial fact that the solution to an LP exists at the boundary of a polytope.

(a) First, consider the case where we have an unbounded polyhedron. For any such \(\mathcal{P}\), assume there exists some \(\vec{x}_0, \vec{v} \in \mathbb{R}^n\) with \(\vec{x}_0 \neq \vec{v}\) such that \(L := \{\vec{x}_0 + \alpha \vec{v} \mid \alpha \in [0, \infty)\} \subset \mathcal{P}\). This is saying that there exists a line segment, unbounded on one side, that is contained within \(\mathcal{P}\). Show that if \(\vec{c}^\top \vec{v} < 0\), then \(p^* = -\infty\).

(b) Now, suppose our feasible set is defined by a bounded polytope (i.e. a polytope that contains its boundary). We say a point belongs to the interior of \(\mathcal{P}\) (i.e. \(\vec{x}_0 \in \text{Int}(\mathcal{P})\)) if there exists some ball of radius \(\epsilon > 0\) such that \(B := \{\vec{y} \in \mathbb{R}^n \mid \|\vec{y} - \vec{x}_0\|_2 \leq \epsilon\} \subset \mathcal{P}\). Show that the optimal point for the LP cannot be obtained in the interior of \(\mathcal{P}\) and must be obtained on the boundary (that is, when for some \(1 \leq i \leq n\), \(\vec{a}_i^\top \vec{x} - b_i = 0\)). For this, consider a proof by contradiction. Show that for any \(x_0\) on the interior, there exists another point

\[
\vec{x}_1 := \vec{x}_0 - \epsilon \frac{\vec{c}}{\|\vec{c}\|_2},
\]

such that \(\vec{x}_1 \in \mathcal{P}\) and \(\vec{c}^\top \vec{x}_1 < \vec{c}^\top \vec{x}_0\).
3. Duality

Consider the function

\[ f(\bar{x}) = \bar{x}^T A \bar{x} - 2 \bar{b}^T \bar{x}. \]  

(6)

First, we consider the unconstrained optimization problem

\[ p^* = \min_{\bar{x} \in \mathbb{R}^n} f(\bar{x}) = \min_{\bar{x} \in \mathbb{R}^n} \bar{x}^T A \bar{x} - 2 \bar{b}^T \bar{x} \]  

(7)

for a real \( n \times n \) symmetric matrix \( A \in \mathbb{S}^n \) and \( \bar{b} \in \mathbb{R}^n \). If the problem is unbounded below, then we say \( p^* = -\infty \). Let \( \bar{x}^* \) denote the minimizing argument of the optimization problem.

(a) Suppose \( A \succeq 0 \) (positive semidefinite) and \( \bar{b} \in \mathcal{R}(A) \). Let \( \text{rk}(A) = n \). Find \( p^* \).

**HINT:** What does \( A \succeq 0 \) tell you about the function \( f \)? How can you leverage the rank of \( A \) to compute \( p^* \)?

(b) Suppose \( A \succeq 0 \) (positive semidefinite) and \( \bar{b} \in \mathcal{R}(A) \) as before. Let \( A \) be rank-deficient, i.e., \( \text{rk}(A) = r < n \). Let \( A \) have the compact/thin and full SVD as follows, with diagonal positive definite \( \Lambda_r \in \mathbb{R}^{r \times r} \):

\[ A = U_r \Lambda_r U_r^T = [U_r \quad U_1] \begin{bmatrix} \Lambda_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_r^T \\ U_1^T \end{bmatrix}. \]  

(8)

Show that the minimizer \( \bar{x}^* \) of the optimization problem (7) is not unique by finding a general form for the family of solutions for \( \bar{x}^* \) in terms of \( U_r, U_1, \Lambda_r, \bar{b} \).

**HINT:** As before, \( A \succeq 0 \) gives you some information about the objective function \( f \). Can you use this information along with the fact that \( b \in \mathcal{R}(A) \) to obtain a general form for the minimizers of \( f \)? Use the fact that any vector \( \bar{x} \in \mathbb{R}^n \) can be written as \( \bar{x} = U_r \bar{\alpha} + U_1 \bar{\beta} \) for unique \( \bar{\alpha}, \bar{\beta} \).

(c) If \( A \not\succeq 0 \) (\( A \) not positive semi-definite) show that \( p^* = -\infty \) by finding \( \bar{v} \) such that \( f(\alpha \bar{v}) \to -\infty \) as \( \alpha \to \infty \).

**HINT:** \( A \not\succeq 0 \) means that there exists \( \bar{v} \) such that \( \bar{v}^T A \bar{v} < 0 \).

(d) Suppose \( A \succeq 0 \) (positive semidefinite) and \( \bar{b} \notin \mathcal{R}(A) \). Find \( p^* \). Justify your answer mathematically.

**HINT:** From FTLA, we know that \( \mathbb{R}^n = \mathcal{R}(A^T) \oplus \mathcal{N}(A) \). Therefore, \( \bar{b} = \bar{v}_1 + \bar{v}_2 \) where \( \bar{v}_1 \in \mathcal{R}(A) = \mathcal{R}(A^T) \) and \( \bar{v}_2 \in \mathcal{N}(A) \).

For parts (e) and (f), consider real \( n \times n \) symmetric matrix \( A \in \mathbb{S}^n \) and \( \bar{b} \in \mathbb{R}^n \). Let \( \text{rk}(A) = r \), where \( 0 \leq r \leq n \). Now we consider the constrained optimization problem

\[ p^* = \min_{\bar{x} \in \mathbb{R}^n} \bar{x}^T A \bar{x} - 2 \bar{b}^T \bar{x} \]  

s.t. \( \bar{x}^T \bar{x} \geq 1 \).  

(9)

(e) Write the Lagrangian \( \mathcal{L}(\bar{x}, \lambda) \), where \( \lambda \) is the dual variable corresponding to the inequality constraint.

(f) For any matrix \( C \in \mathbb{R}^{n \times n} \) with \( \text{rk}(C) = r \leq n \) and compact SVD

\[ C = U_r \Lambda_r V_r^T, \]  

(10)
we define the pseudoinverse as
\[ C^\dagger = V_r \Lambda_r^{-1} U_r^\top. \quad (11) \]

We use the "dagger" operator to represent this. If \( \vec{d} \) lies in the range of \( C \), then a solution to the equation \( C\vec{x} = \vec{d} \), can be written as \( \vec{x} = C^\dagger \vec{d} \), even when \( C \) is not full rank. Show that the dual problem to the primal problem (9) can be written as,
\[
d^* = \max_{\lambda \geq 0, \vec{b} \in \mathbb{R}(A - \lambda I)} -\vec{b}^\top (A - \lambda I)^\dagger \vec{b} + \lambda. \quad (12)\]

**HINT:** To show this, first argue that when the constraints are not satisfied \( \min_\vec{x} \mathcal{L}(\vec{x}, \lambda) = -\infty \). Then show that when the constraints are satisfied, \( \min_\vec{x} \mathcal{L}(\vec{x}, \lambda) = -\vec{b}^\top (A - \lambda I)^\dagger \vec{b} + \lambda \).

**HINT:** Compute \( g(\lambda) \) and explore its behavior under the constraints.
4. A Slalom Problem

A skier must slide from left to right by going through \( n \) parallel gates of known position \((x_i, y_i)\) and width \( c_i \), \( i = 1, \ldots, n \). The initial position \((x_0, y_0)\) is given, as well as the final one, \((x_{n+1}, y_{n+1})\). Before reaching the final position, the skier must go through gate \( i \) by passing between the points \((x_i, y_i - c_i/2)\) and \((x_i, y_i + c_i/2)\) for each \( i \in \{1, \ldots, n\} \).

Figure 1 is an example and does not have the right value of \( n \) nor show the true \((x_i, y_i, c_i)\) values. Use values for \((x_i, y_i, c_i)\) from Table 1.

![Slalom problem](image)

**Figure 1:** Slalom problem with \( n = 6 \) gates. The initial and final positions are fixed and not included in the figure. The skier slides from left to right. The middle path is dashed and connects the center points of gates.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i )</th>
<th>( y_i )</th>
<th>( c_i )</th>
</tr>
</thead>
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<tr>
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<td>0</td>
<td>4</td>
<td>N/A</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>4</td>
<td>2</td>
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<td>12</td>
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</tr>
<tr>
<td>5</td>
<td>20</td>
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<td>2</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
<td>4</td>
<td>N/A</td>
</tr>
</tbody>
</table>

(a) Given the data \( \{(x_i, y_i, c_i)\}_{i=0}^{n+1} \), write an optimization problem that minimizes the total length of the path. Your answer should come in the form of an SOCP.

(b) Solve the problem numerically with the data given in Table 1. **HINT:** You should be able to use packages such as cvxpy and numpy.
5. Sphere Enclosure

Let $B_i, i = 1, \ldots, m$, be $m$ Euclidean balls in $\mathbb{R}^n$, with centers $\vec{x}_i$, and radii $\rho_i \geq 0$. We wish to find a ball $B$ of minimum radius that contains all the $B_i, i = 1, \ldots, m$. Cast this problem as an SOCP.
6. Dual Norms and SOCP

Consider the problem

\[ p^* = \min_{\mathbf{x} \in \mathbb{R}^n} \| A\mathbf{x} - \mathbf{y} \|_1 + \mu \| \mathbf{x} \|_2, \]  

where \( A \in \mathbb{R}^{m \times n}, \mathbf{y} \in \mathbb{R}^m, \text{ and } \mu > 0. \)

(a) Express this (primal) problem in standard SOCP form.

(b) Find a dual to the problem and express it in standard SOCP form. \textit{HINT: Recall that for every vector} \( \mathbf{z} \), the following dual norm equalities hold:

\[
\| \mathbf{z} \|_2 = \max_{\mathbf{u} : \| \mathbf{u} \|_2 \leq 1} \mathbf{u}^\top \mathbf{z}, \quad \| \mathbf{z} \|_1 = \max_{\mathbf{u} : \| \mathbf{u} \|_\infty \leq 1} \mathbf{u}^\top \mathbf{z}.
\]  

(c) Assume strong duality holds\(^1\) and that \( m = 100 \) and \( n = 10^6, \) i.e., \( A \) is \( 100 \times 10^6 \). Which problem would you choose to solve using a numerical solver: the primal or the dual? Justify your answer.

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\(^1\)In fact, you can show that strong duality holds using Sion’s theorem, a generalization of the minimax theorem that is beyond the scope of this class.
7. Homework Process

With whom did you work on this homework? List the names and SIDs of your group members.

**NOTE:** If you didn’t work with anyone, you can put “none” as your answer.