EECS 127/227AT Optimization Models in Engineering UC Berkeley Spring 2024

Homework 11

This homework is due at 11 PM on April 12, 2024.

Submission Format: Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned), as well as a printout of your completed Jupyter notebook(s).

1. **Project Logistics**

   Fill out this form to let us know whether:

   - you plan on doing the project in a group;
   - you plan on doing the project by yourself; or
   - you are not planning to do the project;

   as well as some auxiliary information. Even if you do not plan to do the project, you **must** fill out the form.

   To get credit for the problem, please write down the secret word in the confirmation message displayed after completing the survey.

2. Linear Programming

Express the following problem as an LP.

$$\min_{\vec{x}\in\mathbb{R}^k} \quad \sum_{i=1}^{k} |x_i| \tag{1}$$

$$\text{s.t.} \quad A\vec{x} = \vec{b} \tag{2}$$

3. Fun with Hyperplanes

In this problem we work with hyperplanes, which are key components of linear programming as well as future topics such as support vector machines.

(a) Sketch the hyperplane $\mathcal{H} \doteq \{\vec{x} \in \mathbb{R}^2 \mid \begin{bmatrix} 1 & 1 \end{bmatrix} \vec{x} = 2\}$.

(b) Let $\vec{c} \in \mathbb{R}^n$ be nonzero, and let $\mathcal{H} \doteq \{\vec{x} \in \mathbb{R}^n \mid \vec{c}^\top \vec{x} = 0\}$. Show that $\mathcal{H}$ is a linear subspace of $\mathbb{R}^n$. What is $\dim(\mathcal{H})$?

(c) Let $\vec{c} \in \mathbb{R}^n$ be nonzero, and let $\mathcal{H} \doteq \{\vec{x} \in \mathbb{R}^n \mid \vec{c}^\top \vec{x} = 0\}$. Suppose $\vec{x}_\star \in \mathbb{R}^n$ is on one side of the hyperplane, i.e., $\vec{c}^\top \vec{x}_\star > 0$. Give any vector which is on the other side of the hyperplane but not on the hyperplane itself.

(d) Let $\vec{c} \in \mathbb{R}^n$ be nonzero, and let $\vec{x}_0 \in \mathbb{R}^n$ be arbitrary. Let $\mathcal{H} \doteq \{\vec{x} \in \mathbb{R}^n \mid \vec{c}^\top (\vec{x} - \vec{x}_0) = 0\}$. Suppose $\vec{x}_\star \in \mathbb{R}^n$ is on one side of the hyperplane. Give any vector which is on the other side of the hyperplane but not on the hyperplane itself.

(e) Let $\vec{x}_0 \in \mathbb{R}^n$ be arbitrary. For a vector $\vec{c} \in \mathbb{R}^n$, let $\mathcal{H}(\vec{c}) \doteq \{\vec{x} \in \mathbb{R}^n \mid \vec{c}^\top (\vec{x} - \vec{x}_0) = 0\}$. Show that $\vec{0} \in \mathcal{H}(\vec{c})$ for every $\vec{c} \in \mathbb{R}^n$ if and only if $\vec{x}_0 = \vec{0}$.

4. Formulating problems as LPs or QPs

Formulate the problem

$$p_j^* \doteq \min_{\vec{x}} \; f_j(\vec{x}), \tag{3}$$

for different functions f_j, $j = 1, \ldots, 4$, as convex **QPs** or **LPs**, or, if you cannot, explain why. In our formulations, we always use $\vec{x} \in \mathbb{R}^n$ as the variable, and assume that $A \in \mathbb{R}^{m \times n}$, $\vec{y} \in \mathbb{R}^m$. If you obtain a convex **LP** or **QP** formulation, state precisely what the variables, objective, and constraints are.

(a) $f_1(\vec{x}) = \|A\vec{x} - \vec{y}\|_\infty + \|\vec{x}\|_1$.

(b) $f_2(\vec{x}) = \|A\vec{x} - \vec{y}\|_2^2 + \|\vec{x}\|_1$.

(c) $f_3(\vec{x}) = \|A\vec{x} - \vec{y}\|_2^2 - \|\vec{x}\|_1$.

(d) $f_4(\vec{x}) = \|A\vec{x} - \vec{y}\|_2^2 + \|\vec{x}\|_1^2$.

5. A Slalom Problem

A skier must slide from left to right by going through n parallel gates of known position (x_i, y_i) and width c_i, $i = 1, \ldots, n$. The initial position (x_0, y_0) is given, as well as the final one, (x_{n+1}, y_{n+1}). Before reaching the final position, the skier must go through gate i by passing between the points $(x_i, y_i - c_i/2)$ and $(x_i, y_i + c_i/2)$ for each $i \in \{1, \ldots, n\}$.

Figure 1 is an example and does not have the right value of n nor show the true (x_i, y_i, c_i) values. Use values for (x_i, y_i, c_i) from Table 1.

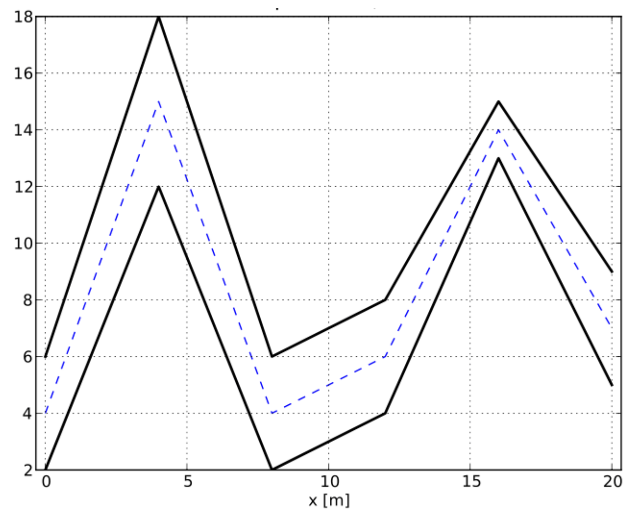

Figure 1: Slalom problem with $n = 6$ gates. The initial and final positions are fixed and not included in the figure. The skier slides from left to right. The middle path is dashed and connects the center points of gates.

Table 1: Problem data for Problem 2. Here $n = 5$.

i	x_i	y_i	c_i
0	0	4	N/A
1	4	5	3
2	8	4	2
3	12	6	2
4	16	8	1
5	20	7	2
6	24	4	N/A

(a) Given the data $\{(x_i, y_i, c_i)\}_{i=0}^{n+1}$, write an optimization problem that minimizes the total length of the path. Your answer should come in the form of an SOCP.

(b) Solve the problem numerically with the data given in Table 1. *HINT: You should be able to use packages such as* `cvxpy` *and* `numpy`*.*

6. Dual Norms and SOCP

Consider the problem

$$p^\star = \min_{\vec{x} \in \mathbb{R}^n} \|A\vec{x} - \vec{y}\|_1 + \mu \|\vec{x}\|_2 \,, \tag{4}$$

where $A \in \mathbb{R}^{m \times n}$, $\vec{y} \in \mathbb{R}^m$, and $\mu > 0$.

(a) Express this (primal) problem in standard SOCP form. *HINT: you can make the objective function linear by introducing slack variables.*

(b) Find a dual to the problem and express it in standard SOCP form. *HINT: You may find Problem 1 in Discussion 11 to be helpful. Recall that for every vector $\vec{z}$, the following dual norm equalities hold:*

$$\|\vec{z}\|_2 = \max_{\vec{u} \,:\, \|\vec{u}\|_2 \leq 1} \vec{u}^\top \vec{z}, \qquad \|\vec{z}\|_1 = \max_{\vec{u} \,:\, \|\vec{u}\|_\infty \leq 1} \vec{u}^\top \vec{z}. \tag{5}$$

Additionally, you may use the following fact: If the primal problem is expressed as $p^\star = \min_{\vec{x}} \max_{\vec{y}} L(\vec{x}, \vec{y})$ (which is $p^\star = \min_{\vec{x}} f_0(x)$ for $f_0(x) = \max_{\vec{y}} L(\vec{x}, \vec{y})$), then the dual problem can be obtained by swapping the max and min: $d^\star = \max_{\vec{y}} \min_{\vec{x}} L(\vec{x}, \vec{y})$.

(c) Assume strong duality holds[1] and that $m = 100$ and $n = 10^6$, i.e., A is 100×10^6. Which problem would you choose to solve using a numerical solver: the primal or the dual? Justify your answer.

[1] In fact, you can show that strong duality holds using Sion's theorem, a generalization of the minimax theorem that is beyond the scope of this class.

7. Homework Process

With whom did you work on this homework? List the names and SIDs of your group members.

NOTE: If you didn't work with anyone, you can put "none" as your answer.