EECS 127/227AT Optimization Models in Engineering

Spring 2019 Final

Final exam

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ________________, ________________ ________________

(last name) (first name) (signature)

Name and SID of the person to your left: ________________________________

Name and SID of the person to your right: ________________________________

Name and SID of the person in front of you: ________________________________

Name and SID of the person behind you: ________________________________

1. (1 Point) Tell us about a time that you succeeded this semester.

2. (1 Point) What are you looking forward to over summer break?

Do not turn this page until the proctor tells you to do so. You may work on the questions above.

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Extra page for scratchwork.

Work on this page will NOT be graded.

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3. (3 points) Convexity of functions

Let $g : \mathbb{R}^m \to \mathbb{R}$ be **any** convex function. For any $A \in \mathbb{R}^{m \times n}$, prove that $f : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(x) = g(Ax) \quad \forall x \in \mathbb{R}^n,$$

is also a convex function.

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4. (2 points) Multiple Choice

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Consider the following optimization problems:

$$\begin{aligned} p_1^* = \min_{t \in \mathbb{R}, x \in \mathbb{R}^n} \quad & t \\ \text{s.t.} \ & \|x\|_2 = t, \\ & f(x) \le 0, \end{aligned} \tag{1}$$

$$\begin{aligned} p_2^* = \min_{t \in \mathbb{R}, x \in \mathbb{R}^n} \quad & t \\ \text{s.t.} \ & \|x\|_2 \le t, \\ & f(x) \le 0. \end{aligned} \tag{2}$$

Write the statement labels (A, B, C) corresponding to statements that are true in the box given below. More than one statement might be true; and you will get credit for this problem only if you write the labels corresponding to all statements that are true and do not write a label corresponding to any statement that is false. No justification is required.

(A) Problem (1) as written is a convex problem.

(B) Problem (2) as written is a convex problem.

(C) We necessarily have $p_1^* = p_2^*$.

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5. (5 points) Connecting all Bears

A cellular service provider, BearT&T™, wants to place a base station in Berkeley so as to maximize the quality of service provided to its customers. Let $z_1, z_2, \ldots, z_m \in \mathbb{R}^2$ denote the fixed locations of the customers. The location for the new base station is given by the solution to:

$$\min_{x \in \mathbb{R}^2} \max_{i \in \{1,\ldots,m\}} \|x - z_i\|_2. \tag{3}$$

(a) (2 points) Explain why (3) is a convex problem.

(b) (3 points) Cast this problem in one of the standard convex optimization problems we have seen in class: LP, QP, QCQP or SOCP.

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6. (3 points) Newton's method

Given a symmetric positive definite matrix $Q \in \mathbb{S}^n_{++}$ and $b \in \mathbb{R}^n$, consider the minimization of the function $f : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(x) = \frac{1}{2} x^\top Q x - b^\top x.$$

Let x^* denote the point at which $f(x)$ is minimized, and define $\mathcal{B}(x^*)$ as the ball centered at x^* with unit ℓ_2-norm:

$$\mathcal{B}(x^*) = \{x \in \mathbb{R}^n : \|x - x^*\|_2 \leq 1\}.$$

Assume we use Newton's method to minimize f:

$$x_{k+1} = x_k - (\nabla^2 f(x_k))^{-1} \nabla f(x_k),$$

where the initial point is $x_0 \in \mathcal{B}(x^*)$. For any $k \in \mathbb{N}$, find

$$\max_{x_0 \in \mathcal{B}(x^*)} \|x_k - x^*\|_2.$$

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7. (11 points) Linear algebra meets optimization

Let wide matrix $A \in \mathbb{R}^{m\times n}$ ($m < n$) be full row rank.

(a) (2 points) Consider the ridge regression problem, where $b \in \mathbb{R}^m, x \in \mathbb{R}^n$ and the constant $\lambda > 0$ is given:

$$\min_x \quad ||Ax - b||_2^2 + \lambda||x||_2^2 \tag{4}$$

Since this is a convex problem and the objective function is differentiable, the optimum can be found by setting the gradient to zero. Use this to find the optimal solution x^*.

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(b) (6 points) Now we rewrite the problem in (4) by adding a constraint

$$\min_{z=Ax-b} ||z||_2^2 + \lambda||x||_2^2. \tag{5}$$

Let the Lagrangian corresponding to this problem be $\mathcal{L}(x, z, \nu)$, where ν is the dual variable corresponding to the equality constraint. Write out the dual function $g(\nu) = \inf_{x,z} \mathcal{L}(x, z, \nu)$ explicitly. Solve the dual problem to get ν^*. Find the corresponding values of $\tilde{x}, \tilde{z}$ such that $g(\nu^*) = \mathcal{L}(\tilde{x}, \tilde{z}, \nu^*)$.

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(c) (3 points) Show that for every $\lambda > 0$,

$$(A^\top A + \lambda I)^{-1} A^\top b = A^\top (AA^\top + \lambda I)^{-1} b.$$

Hint: One approach is to start by considering $\lambda A^\top + A^\top A A^\top$. Another approach is to use the SVD of A.

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8. (9 points) A matrix optimization problem

Consider the following optimization problem

$$\begin{aligned} \min_{X \in \mathbb{R}^{n \times n}} \quad & \frac{1}{2} \|X\|_F^2 \\ \text{s.t.} \quad & X \in \mathcal{S}, \end{aligned}$$

where $\mathcal{S} = \{A \in \mathbb{R}^{n \times n} \mid \sigma_{\min}(A) \geq 2\}$ and $\sigma_{\min}(A)$ refers to the smallest singular value of A.

(a) (2 points) Is the objective function convex? Justify.

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(b) (3 points) Is the constraint set convex? Justify.

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(c) (4 points) By using the singular value decomposition of X, rewrite the objective function and constraints in terms of the singular values of X and find a solution X^*.

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9. (8 points + 5 bonus points) Energy functions for linear systems

Given $A \in \mathbb{R}^{n \times n}$ and $x_0 \in \mathbb{R}^n$, consider the linear-time-invariant system with no input:

$$x_{k+1} = Ax_k \quad \forall k \in \mathbb{N}. \tag{6}$$

Let P be a symmetric positive semidefinite matrix, and let Q be a symmetric positive definite matrix in $\mathbb{R}^{n \times n}$; that is, $P \in \mathbb{S}^n_+$ and $Q \in \mathbb{S}^n_{++}$. Assume P and Q satisfy

$$A^\top PA - P \preceq -Q, \tag{7}$$

and define $V : \mathbb{R}^n \to \mathbb{R}$ as

$$V(x_k) = x_k^\top P x_k \quad \forall x_k \in \mathbb{R}^n.$$

(a) (2 points) Show that $V(x_{k+1}) - V(x_k) \leq 0$ for all $k \in \mathbb{N}$.
Hint: Use (7).

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(b) (4 points) Find a constant $\beta \in (0, \infty)$ in terms of eigenvalues of P and Q such that

$$V(x_{k+1}) \leq (1 - \beta) V(x_k) \quad \forall k \in \mathbb{N}.$$

Hint: If you can find $\alpha, \gamma \in (0, \infty)$ *such that*

$$x_k^\top Q x_k \geq \alpha \|x_k\|_2^2 \quad \forall x_k \in \mathbb{R}^n,$$

$$x_k^\top P x_k \leq \gamma \|x_k\|_2^2 \quad \forall x_k \in \mathbb{R}^n,$$

then

$$-x_k^\top Q x_k \leq -\frac{\alpha}{\gamma} x_k^\top P x_k = -\frac{\alpha}{\gamma} V(x_k) \quad \forall x_k \in \mathbb{R}^n.$$

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(c) (2 points) Along with the nonnegativity of $V(x_k)$, part (b) shows that $V(x_k)$ converges to zero for every initialization. In analysis of dynamical systems, functions like V are used to represent the energy in the system, and its convergence to zero indicates the dissipation of this energy. For this reason, finding V is an important problem in the study of dynamical systems.

Given $A \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{S}^n_{++}$ for system (6), we can use the following optimization problem to find the matrix P defining the function V:

$$\begin{aligned} \min_{P\in\mathbb{S}^n_+} \quad & 0 \\ \text{subject to} \quad & A^\top PA - P + Q \preceq 0. \end{aligned} \tag{8}$$

Only for this part of the question, consider the scalar version of problem (8):

$$\begin{aligned} \min_{p\in\mathbb{R}} \quad & 0 \\ \text{subject to} \quad & p \geq 0 \\ & a^2p - p + q \leq 0, \end{aligned} \tag{9}$$

where $a, q \in \mathbb{R}$ are some fixed constants and $q > 0$. Under what conditions on a and q, is the problem (9) feasible?

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(d) (**Bonus**: 5 points) Consider the non-scalar problem (8) again. Find the dual problem corresponding to (8) by explicitly deriving the dual function and the feasibility constraints of the dual problem. Do **not** dualize the constraint $P \in \mathbb{S}^n_+$.

Hint: For any $Y \in \mathbb{S}^n$*, we have*

$$\max_{\Lambda \succeq 0} \langle Y, \Lambda \rangle = \begin{cases} 0 & \textit{if } Y \preceq 0 \\ +\infty & \textit{otherwise.} \end{cases}$$

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10. (6 points) Minimizing quadratics

Consider the following optimization problem:

$$p^* = \inf_{x \in \mathbb{R}^2} x^\top A x + b^\top x,$$

where $A \in \mathbb{S}^2_+$ and $b \in \mathbb{R}^2$.

(a) (3 points) Suppose $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Find a vector b with $\|b\|_2 = 1$ such that $p^* > -\infty$.

Hint: Is A invertible?

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(b) (3 points) Now assume A is a symmetric positive definite matrix, i.e. $A \in \mathbb{S}^2_{++}$ and $b = [0, 0]^\top$. Suppose we add a ℓ_∞-norm regularizer term to the objective to get the following optimization problem:

$$p^* = \inf_{x \in \mathbb{R}^2} x^\top A x + \|x\|_\infty .$$

Write the corresponding dual problem as

$$\begin{aligned} d^* = \sup_{y \in \mathbb{R}^2} \quad & g(y) \\ \text{subject to} \quad & \|y\|_c \leq 1, \end{aligned}$$

where you will determine $g(y)$ and c.
Hint: For every $x \in \mathbb{R}^2$, we have

$$\sup_{y \in \mathbb{R}^2 \,:\, \|y\|_1 \leq 1} x^\top y = \|x\|_\infty .$$

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11. (13 points) A matrix game

Let $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ be a payoff matrix for two games as described in the parts below. Suppose row player, R, chooses action x and column player, C, chooses action y, then both players get the payoff $s = x^\top Ay$. R wishes to minimize payoff, while C wishes to maximize payoff.

(a) Suppose $x \in \mathcal{E}, y \in \mathcal{E}$, where $\mathcal{E} = \{[0,1]^\top, [1,0]^\top\}$.

i. (3 points) Suppose R chooses x first and then C chooses y. The optimal payoff s_R^* is given by

$$s_R^* = \min_{x \in \mathcal{E}} \max_{y \in \mathcal{E}} x^\top Ay.$$

For the given matrix A, $s_R^* = 3$ achieved for $x^* = [1,0]^\top, y^* = [0,1]^\top$.

Now suppose C chooses y first and then R chooses x. The optimal payoff s_C^* is given by,

$$s_C^* = \max_{y \in \mathcal{E}} \min_{x \in \mathcal{E}} x^\top Ay.$$

Find s_C^* for the given matrix A. Justify your answer.

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ii. (1 point) Compare s_R^* to s_C^*. Who is better off — the first player or the second player?

iii. (2 points) Now suppose A was unknown. Does your choice of whether to go first or second remain the same? Justify.

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(b) Suppose $x \in \mathcal{P}, y \in \mathcal{P}$ where $\mathcal{P} = \{z = [z_1, z_2]^\top \in \mathbb{R}^2 \mid z_1 \geq 0, z_2 \geq 0, z_1 + z_2 = 1\}$. Suppose R chooses x first and then C chooses y. Let p_R^* denote the optimal payoff in this case given by,

$$p_R^* = \min_{x \in \mathcal{P}} \max_{y \in \mathcal{P}} \; x^\top A y.$$

i. (3 points) For a given $x \in \mathcal{P}$, show that $\max\limits_{y \in \mathcal{P}} x^\top A y = \max\limits_{y \in \mathcal{E}} x^\top A y$.

Hint: Show that

$$\max_{y \in \mathcal{P}} \; x^\top A y \leq \max_{y \in \mathcal{E}} \; x^\top A y,$$
$$\max_{y \in \mathcal{P}} \; x^\top A y \geq \max_{y \in \mathcal{E}} \; x^\top A y.$$

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ii. (4 points) Formulate a Linear Program with finitely many constraints to find p_R^*, which is equivalent to

$$p_R^* = \min_{x \in \mathcal{P}} \max_{y \in \mathcal{E}} \ x^\top A y$$

due to result of part (i).

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12. (5 points + 3 bonus points) Soft-margin SVM

Consider the soft-margin SVM problem,

$$p^*(C) = \min_{w\in\mathbb{R}^m, b\in\mathbb{R}, \xi\in\mathbb{R}^n} \frac{1}{2}\|w\|_2^2 + C\sum_{i=1}^{n}\xi_i \tag{10}$$
$$\text{s.t. } 1-\xi_i - y_i(x_i^\top w - b) \le 0, \quad i=1,2,\dots,n$$
$$-\xi_i \le 0, \quad i = 1,2,\dots,n,$$

where $x_i \in \mathbb{R}^m$ refers to the i^{th} training data point, $y_i \in \{-1, 1\}$ is its label, and $C \in \mathbb{R}_+$ (i.e. $C > 0$) is a hyperparameter.

Let α_i denote the dual variable corresponding to the inequality $1 - \xi_i - y_i(x_i^\top w - b) \le 0$ and let β_i denote the dual variable corresponding to the inequality $-\xi_i \le 0$.

The Lagrangian is then given by

$$\mathcal{L}(w, b, \xi, \alpha, \beta) = \frac{1}{2}\|w\|_2^2 + C\sum_{i=1}^{n}\xi_i + \sum_{i=1}^{n}\alpha_i(1-\xi_i - y_i(x_i^\top w - b)) - \sum_{i=1}^{n}\beta_i\xi_i.$$

Suppose $w^*, b^*, \xi^*, \alpha^*, \beta^*$ satisfy the KKT conditions.

Classify the following statements as true or false. **Justify your answers mathematically**. A correct answer with missing or incorrect justification will be given 0 points.

(a) (3 points) Suppose the optimal solution w^*, b^* changes when the training point x_i is removed. Then originally, we necessarily have $y_i(x_i^\top w^* - b^*) = 1 - \xi_i^*$.

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(b) (2 points) Suppose the optimal solution w^*, b^* changes when the training point x_i is removed. Then originally, we necessarily have $\alpha_i^* > 0$.

(c) (**Bonus**: 3 points) Suppose the data points are strictly linearly separable, i.e. there exist $\tilde{w}$ and $\tilde{b}$ such that for all i,

$$y_i(x_i^\top \tilde{w} - \tilde{b}) > 0.$$

Then $p^*(C) \to \infty$ as $C \to \infty$.

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Extra page for scratchwork.

Work on this page will NOT be graded.

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Extra page for scratchwork.
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Doodle page!
Draw us something if you want or give us suggestions, compliments, or complaints.
You can also use this page to report anything suspicious that you might have noticed.

EECS 127/227AT Optimization Models in Engineering

Spring 2019 Final Instructions

Read the following instructions before the exam.

There are 12 problems of varying numbers of points. You have 180 minutes for the exam. The problems are of varying difficulty, so pace yourself accordingly, do easier problems first, and avoid spending too much time on any one question until you have gotten all of the other points you can. This exam is out of 67 points. There are 8 bonus points on the exam, for a total of 75 points. The 8 points serve as extra credit only, so it is possible to get a perfect score without attempting them.

There are 28 pages on the exam, so there should be 14 sheets of paper in the exam. The exam is printed double-sided. Do not forget the problems on the back sides of the pages! Notify a proctor immediately if a page is missing. **Do not tear out or remove any of the pages. Do not remove the exam from the exam room.**

In the interest of fairness, we will not be able to take any questions during the exam.

No collaboration is allowed, and do not attempt to cheat in any way. Cheating will not be tolerated.

Write your student ID on each page. If a page is found without a student ID, we are not responsible for identifying the student who wrote that page. You may consult THREE handwritten 8.5” × 11” note sheets (front and back). No phones, calculators, tablets, computers, other electronic devices, or scratch paper are allowed.

Please write your answers legibly in the boxed spaces provided on the exam. The space provided should be adequate.

In general, show all of your work in order to receive full credit.

Partial credit will be given for substantial progress on each problem.

If you need to use the restrooms during the exam, bring your student ID card, your phone, and your exam to a proctor. You can collect them once you return from the restrooms. You will not be allowed to turn in your exam or leave the room ten minutes before the end of the exam.

Good luck!

Do not turn this page until the proctor tells you to do so.