EECS 127/227AT Optimization Models in Engineering

Spring 2020 — Final

Write your Student ID and full name on the first page of your submission. Start every subpart of every problem on a new page, except for the Honor Code and the first two questions, which can all be written on the first page. **The top of every page should clearly state the problem and subpart being solved.**

There are a total of 107 points on this exam.

HONOR CODE
Copy the following statements below and sign your name.

I will respect my classmates and the integrity of this exam by following this honor code. I affirm:

- *I have read the instructions for this exam in the associated Piazza posts and understand them.*
- *All of the work submitted here is my original work.*
- *I did not reference any sources on the internet other than the course textbooks, notes, homework and discussion sheets.*
- *I did not collaborate with any other human being on this exam.*

1. **(2 Points) What are you looking forward to over summer break?**

2. **(2 Points) If you had a superpower, what would it be?**

Start every subpart of every problem on a new page.

3. (8 points) Newton's method
Consider the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = x^4.$$

(a) (2 points) Find the optimal value $x^* = \text{argmin}_x f(x)$.

(b) (6 points) Now, we analyze the performance of Newton's method on this problem. Starting from x_0, for $k \geq 0$ we take Newton steps of the form

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}.$$

Find the minimum number of Newton steps that are required to be within a distance of $\epsilon > 0$ from optimum x^*. You may use the optimum value of x^* from part (a). Formally find $k^* \in \mathbb{N}$ (where $\mathbb{N}$ is the set of positive integers) which is the smallest k for which $|x_k - x^*| \leq \epsilon$, i.e.,

$$k^* = \min_{k \in \mathbb{N} : |x_k - x^*| \leq \epsilon} k.$$

Assume that $x_0 > \epsilon > 0$. Your answer should be in terms of ϵ and x_0.

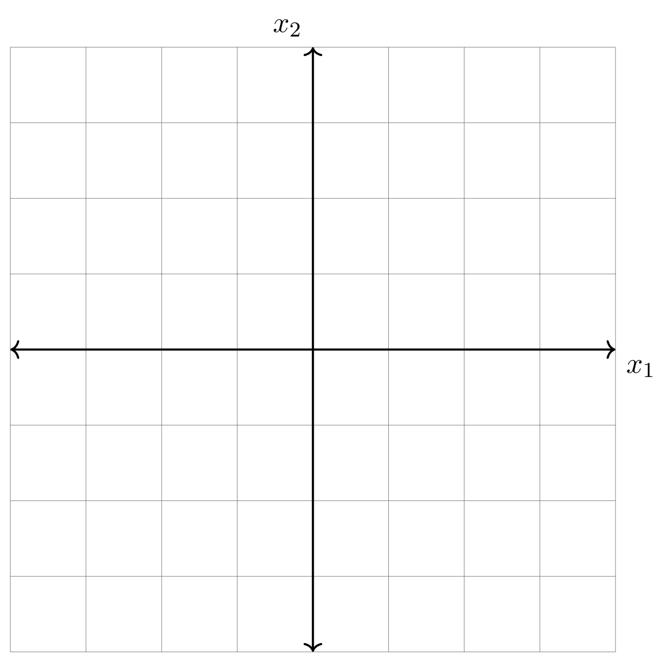
Start every subpart of every problem on a new page.

4. (9 points) A Linear Program

Consider the following LP:

$$
\begin{aligned}
p^* = \min_{\vec{x} \in \mathbb{R}^2} \; & x_1 + x_2 \\
\text{s.t. } & x_1 \geq 0 \\
& x_2 \geq 0 \\
& 1 - x_1 - x_2 \leq 0 \\
& x_1 \leq 3 \\
& x_2 \leq 3.
\end{aligned}
$$

(a) (5 points) Copy the axes below onto your answer sheet. **Plot the feasible region for the optimization problem above. Label your constraints.**



(b) (4 points) **Find** p^*. Justify your answer. *Hint: You don't have to find the dual to solve this part.*

Start every subpart of every problem on a new page.

5. (10 points) Reformulate

(a) (5 points) **Reformulate** the following problem as an SOCP:

$$\min_{\vec{x}} \max_{i=1,2,\ldots,m} \|A\vec{x} - B\vec{y}_i\|_2 .$$

(b) (5 points) The optimization problem below is not a convex QP because A is not a positive semidefinite symmetric matrix. **Reformulate the following problem as a convex QP:**

$$\begin{aligned} &\min_{\vec{x} \in \mathbb{R}^2} \vec{x}^\top A \vec{x} \\ &\text{s.t. } \vec{c}^\top \vec{x} \geq 1, \end{aligned}$$

where $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $\vec{c} \in \mathbb{R}^2$.

Hint: You can write $\vec{x}^\top A\vec{x}$ as $\vec{x}^\top B\vec{x}$, where B is symmetric.

Start every subpart of every problem on a new page.

6. (14 points) Orthogonal lines

(a) (4 points) **Finding the equation of a line.** Suppose we have n noisy samples from a line in the 2D plane $\{(x_i, y_i)\}_{i=1}^n$, which is governed by the equation:

$$\beta x + y = \gamma_1 \qquad \text{(line } A\text{)}.$$

We would like to use the samples to recover the parameters of the line $\vec{z} = [\beta, \gamma_1]^\top$.
We can formulate this as a least squares problem:

$$\min_{\vec{z}} \left\| G\vec{z} - \vec{h} \right\|_2^2,$$

for matrix $G \in \mathbb{R}^{n\times 2}$ and vector $\vec{h} \in \mathbb{R}^n$. **Find G and $\vec{h}$.**

(b) (4 points) Suppose now, we have two lines in the 2D plane:

$$\begin{aligned} \beta x + y &= \gamma_1 \qquad \text{(line } A\text{)} \\ -x + \beta y &= \gamma_2 \qquad \text{(line } B\text{)} \end{aligned}$$

Show that lines A and B are orthogonal to one another.

(c) (6 points) Consider lines A and B as in part (b). Suppose we have n noisy samples from each line: $\{(x_{A_i}, y_{A_i})\}_{i=1}^n$ and $\{(x_{B_i}, y_{B_i})\}_{i=1}^n$, where x_{A_i} represents the x–coordinate of the i^{th} data point generated from line A, and so on.

We are interested in recovering the equations of both lines from these noisy samples. We could estimate the parameters of the two lines independently as in part (a), but then our recovered lines would not necessarily be orthogonal.

To this end, we can write an optimization problem to perform joint estimation of the parameters $\vec{w} = [\beta, \gamma_1, \gamma_2]^\top$:

$$\min_{\vec{w}} \left\| C\vec{w} - \vec{d} \right\|_2^2,$$

for $C \in \mathbb{R}^{2n\times 3}$ and $\vec{d} \in \mathbb{R}^{2n}$. **Find C and $\vec{d}$.**

Start every subpart of every problem on a new page.

7. (11 points) 1D regularization

Consider the following regularized one-dimensional regression problem for $x \in \mathbb{R}$:

$$x^* = \arg\min_x \; f(x),$$

for

$$f(x) = \frac{1}{2}(\alpha - x)^2 + \lambda \mathbb{I}_{x \neq 0}.$$

Here, $\alpha \in \mathbb{R}, \alpha \neq 0$ is known and fixed and

$$\mathbb{I}_{x\neq 0} = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

so the regularizer penalizes the objective function when $x \neq 0$.

(a) (8 points) **Find** x^*. Show your work.

(b) (3 points) **What happens to** x^* **if** $\lambda \to \infty$**?** You may solve this using the previous part or without using it.

Start every subpart of every problem on a new page.

8. (21 points) Minimizing a Quadratic Form

Let $A \in S^n$ and $B \in S^n_{++}$, i.e. A is symmetric and B is positive definite. Consider the following optimization problem:

$$\begin{aligned} p^* = \quad & \min_{\vec{x} \in \mathbb{R}^n} \quad \vec{x}^\top A \vec{x} \\ & \text{s.t.} \quad \vec{x}^\top B \vec{x} \geq 1. \end{aligned}$$

(a) (3 points) **Is this optimization problem convex as stated?** Provide justification.

(b) (6 points) **Find the Lagrangian dual problem using $\lambda \in \mathbb{R}$ as the dual variable corresponding to the constraint.**

It turns out that for this problem strong duality holds.

(c) (6 points) Assume $A \succeq 0$. Let $\vec{x}^* \in \mathbb{R}^n$ and $\lambda^* \geq 0$ be optimal solutions to the primal and dual programs, respectively. **Show that**

$$A\vec{x}^* = \lambda^* B \vec{x}^*.$$

(d) (6 points) Again, assume $A \succeq 0$ and let $\vec{x}^* \in \mathbb{R}^n$ and $\lambda^* \geq 0$ be optimal solutions to the primal and dual programs. **Show that**

$$\vec{x}^* \notin \mathcal{N}(A) \implies \vec{x}^{*\top} B \vec{x}^* = 1.$$

Start every subpart of every problem on a new page.

9. (30 points) Duality

We aim to solve the following problem with convex duality:

$$p^* = \min_{\vec{x} \in \mathbb{R}^d} \|\vec{y} - \vec{x}\|_\infty \quad \text{s.t } \|\vec{x}\|_1 \leq \mu. \tag{Primal}$$

for some $\mu > 0$. Here $\vec{x} \in \mathbb{R}^d$ is the variable we optimize over and $\vec{y} \in \mathbb{R}^d$ is a fixed and known vector. Assume that $\|\vec{y}\|_1 > \mu$ (otherwise we could set $\vec{x} = \vec{y}$ and obtain a optimal value of 0).

(a) (2 points) First, let us consider a simple example, to gain intuition. Let $\vec{y} = [2, 1]^\top$, and let $\mu = 1$. Here $p^* = 1$. **Find the optimal solution $\vec{x}^*$ that achieves this value.**

(b) (2 points) Now, consider a second example. Let $\vec{y} = [2, 1]^\top$, and let $\mu = 2$. Here $p^* = 0.5$. **Find the optimal solution $\vec{x}^*$ that achieves this value.**

Now, assume the entries of $\vec{y}$ are ordered as $y_1 \geq y_2 \cdots \geq y_d \geq 0$ and suppose $y_1 \geq \tau^* \geq 0$ satisfies:

$$\sum_{i=1}^{d} (y_i - \min(y_i, \tau^*)) = \mu.$$

We will show that τ^* is the optimal solution to the problem (**??**), i.e $\tau^* = p^*$ (It can be shown that such a τ^* always exists but you will not be required to prove this).

(c) (6 points) Next define $\vec{w}$ such that,

$$w_i = y_i - \min(y_i, \tau^*).$$

Prove that $\|\vec{y} - \vec{w}\|_\infty = \tau^*$ and $\|\vec{w}\|_1 = \mu$. Justify why this implies that $p^* \leq \tau^*$.

(d) (10 points) Derive a dual of the problem by introducing a vector valued dual variable, $\vec{z}$ for the ℓ_∞ norm in the objective and a scalar valued dual variable, λ, for the constraint. **Formally, prove that a dual of the above program is:**

$$\begin{aligned} \max_{\vec{z}, \lambda} \ & \vec{z}^\top \vec{y} - \lambda \mu \\ \text{s.t } & \|\vec{z}\|_1 \leq 1 \\ & \|\vec{z}\|_\infty \leq \lambda \\ & \lambda \geq 0. \end{aligned} \tag{Dual}$$

Hint: Write out the ℓ_∞ norm in the primal objective as $\max_{\|\vec{z}\|_1 \leq 1} \vec{z}^\top (\vec{y} - \vec{x})$.

Now, let $k = \max\{i : y_i \geq \tau^*\}$. Because the y_i are in decreasing order, we know that $y_1 \geq y_2 \geq \cdots \geq y_k \geq \tau^*$, and k is the number of y_i that are greater than or equal to τ^*. Note that $k > 0$ as $y_1 > \tau^*$.

(e) (8 points) We will now design a dual feasible point, $(\vec{z}^*, \lambda^*)$ with dual objective value τ^* for the dual problem given in (**??**). Consider $\vec{z}^* = [z_1^*, z_2^*, \ldots, z_d^*]^\top$ defined as:

$$z_i^* = \begin{cases} \frac{1}{k} & i = 1, 2, \ldots, k \\ 0 & i = k+1, \ldots, d. \end{cases}$$

- First, **verify that** $\|\vec{z}^*\|_1 \leq 1$**.**
- Next, **find a** λ^* **such that** $(\vec{z}^*, \lambda^*)$ **is feasible for** (??) **and the value of the dual objective is** τ^***.** Highlight your choice for λ^* by drawing a box around it. Justify your answer.

(f) (2 points) Using the results from the previous parts, **justify why** τ^* **is the optimal value of** (??)**.**