Midterm

1. **Honor Code (0 pts)**

   Please copy the following statement in the space provided below and sign your name.

   As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.

   If you do not copy the honor code and sign your name, you will get a 0 on the exam.

Solution:

2. **SID (3 pts)**

   When the exam starts, write your SID at the top of every page. *No extra time will be given for this task.*

3. **Favorites. Any answer, as long as you write it down, will be given full credit. (2 pts)**

   (a) (1 pts) **What's your favorite restaurant in Berkeley?**

   Solution: Any answer is fine.

   (b) (1 pts) **What's some music that makes you happy?**

   Solution: Any answer is fine.

4. Rank 1 Approximation Error (7 pts)

Consider a 2×2 matrix

$$A = \begin{bmatrix} -4 & 8 \\ 7 & 1 \end{bmatrix}. \tag{1}$$

The eigenvalue-eigenvector pairs of $A^\top A$ are

$$(\lambda_1, \vec{v}_1) = \left(90, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\right), \qquad (\lambda_2, \vec{v}_2) = \left(40, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right). \tag{2}$$

(a) (3 pts) **What are the singular values of the matrix A?** *You do not have to justify your answer.*

HINT: You should not have to do much computation here.

Solution: We have that $\sigma_i\{A\} = \sqrt{\lambda_i\{A^\top A\}}$ by definition. Hence $\sigma_1\{A\} = \sqrt{90}$ and $\sigma_2\{A\} = \sqrt{40}$.

(b) (4 pts) For A as given in (1) with $A^\top A$ as given in (2), consider the following problem:

$$p^\star = \min_{\substack{B \in \mathbb{R}^{2\times 2} \\ \mathrm{rank}(B)=1}} \|A - B\|_F^2 . \tag{3}$$

What is the value of $p^\star$? *Justify your answer.*

HINT: You should not have to do any computation at all here.

Solution: The rank 1 approximation minimizing the Frobenius norm error of a matrix $A = \sum_{i=1}^{n} \sigma_i\{A\} u_i v_i^\top$ is $\sigma_1\{A\} u_1 v_1^\top$. Hence the squared Frobenius norm error for the best rank 1 approximation is $\sum_{i=2}^{n} \sigma_i\{A\}^2$. In our case, since the matrix is 2×2, this error just becomes $\sigma_2\{A\}^2 = 40$.

5. ℓ^∞ Constraint (7 pts)

Consider the optimization problem

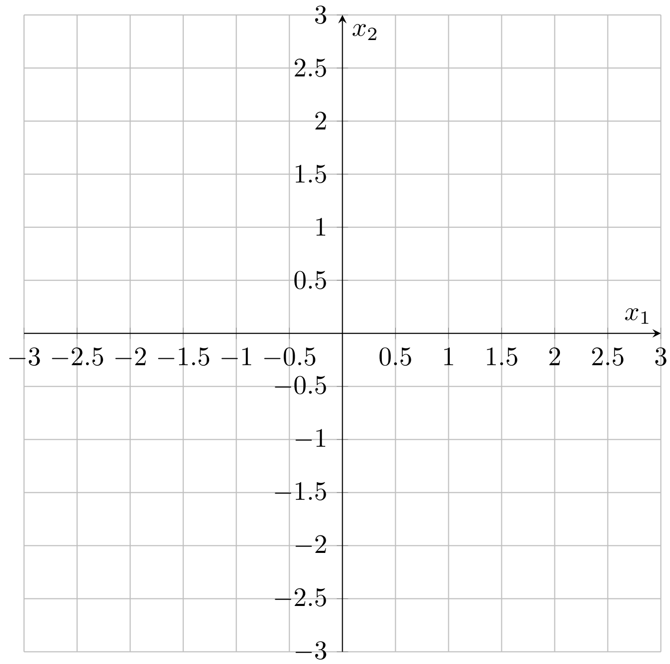
$$\min_{\vec{x} \in \mathbb{R}^2} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}^\top \vec{x} \tag{4}$$

$$\text{s.t.} \quad \|\vec{x}\|_\infty \leq 1 \tag{5}$$

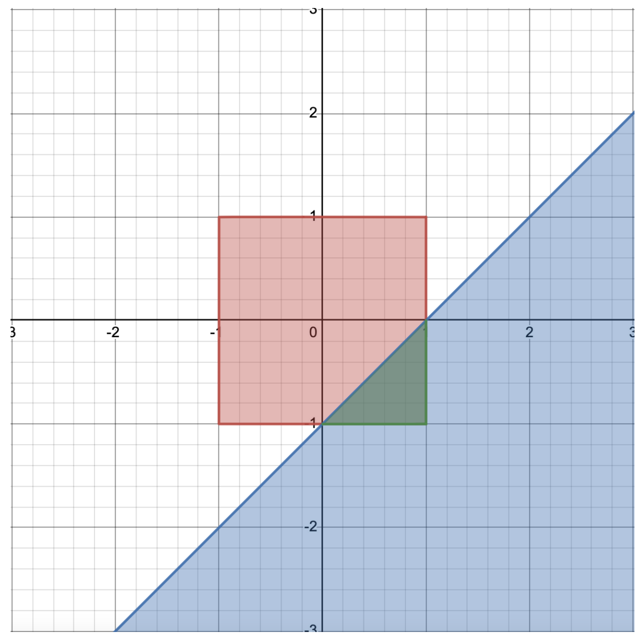
$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}^\top \vec{x} \leq -1, \tag{6}$$

where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(a) (4 pts) **Sketch the feasible region of the above optimization problem in the graph provided below. Label each constraint.**

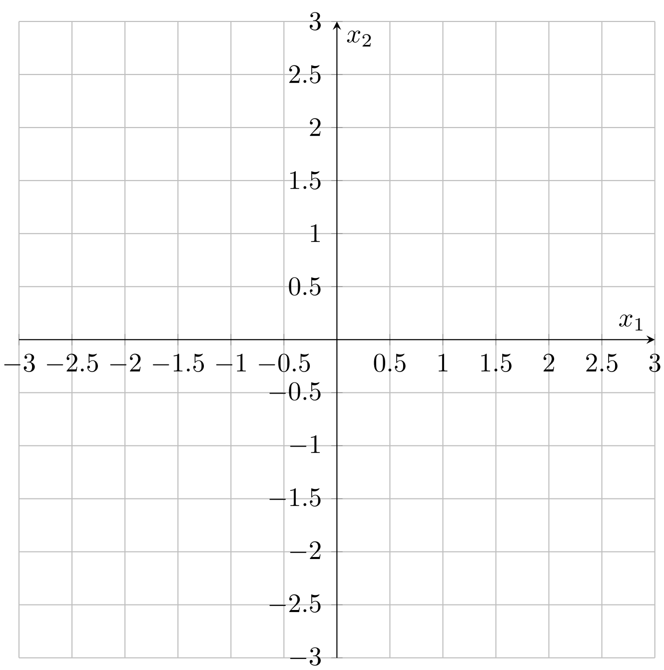


Solution:

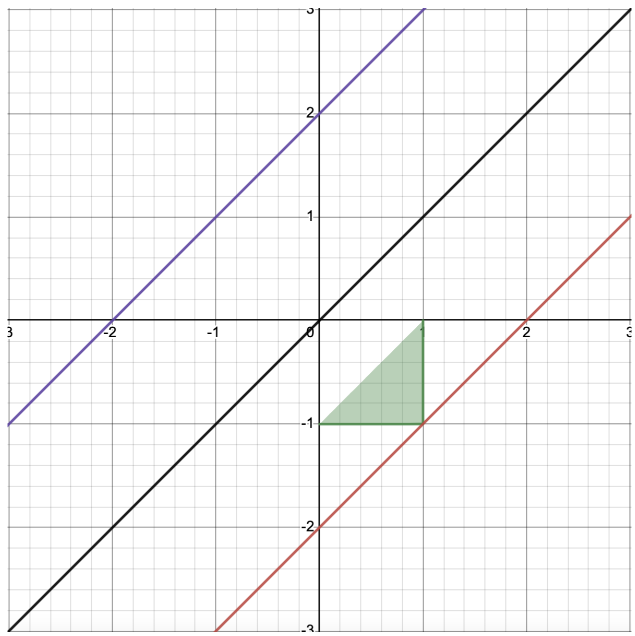


Here the first constraint is in red, the second constraint is in blue, and the overall feasible region is the green polygon.

(b) (3 pts) **Sketch the 2-level set, the 0-level set, and the -2-level set of the objective function in (4). Label each set.**



Solution:



We see here the level sets of the objective function – the -2-level set in purple, the 0-level set in black, and the 2-level set in red.

6. Convexity of the Probability Simplex (5 pts)

Let n be a positive integer. The *probability simplex* on $\mathbb{R}^n$, denoted $\mathcal{P}_n$, is the set

$$\mathcal{P}_n = \left\{ \vec{x} \in \mathbb{R}^n \;\middle|\; x_i \geq 0 \, \forall i, \; \sum_{i=1}^{n} x_i = 1 \right\} \qquad \text{where} \qquad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}. \tag{7}$$

Is $\mathcal{P}_n$ convex? If yes, prove it. If no, justify your answer using an example.

Solution: Yes, $\mathcal{P}_n$ is convex. Let $\vec{x}, \vec{y} \in \mathcal{P}_n$ let $\theta \in [0, 1]$, and define $\vec{z} = \theta\vec{x} + (1-\theta)\vec{y}$. We show that $\vec{z} \in \mathcal{P}_n$. Indeed,

$$\begin{aligned}
z_i &= \underbrace{\theta}_{\geq 0} \underbrace{x_i}_{\geq 0} + \underbrace{(1-\theta)}_{\geq 0} \underbrace{y_i}_{\geq 0} && (8)\\
&\geq 0. && (9)\\
\sum_{i=1}^{n} z_i &= \sum_{i=1}^{n} (\theta x_i + (1-\theta) y_i) && (10)\\
&= \sum_{i=1}^{n} \theta x_i + \sum_{i=1}^{n} (1-\theta) y_i && (11)\\
&= \theta \underbrace{\sum_{i=1}^{n} x_i}_{=1} + (1-\theta) \underbrace{\sum_{i=1}^{n} y_i}_{=1} && (12)\\
&= \theta + (1-\theta) = 1. && (13)
\end{aligned}$$

Thus $\vec{z} \in \mathcal{P}_n$ so $\mathcal{P}_n$ is convex.

7. Vector Calculus (12 pts)

(a) (6 pts) Let $A \in \mathbb{S}^n$ be an $n \times n$ symmetric matrix. **Compute the gradient with respect to $\vec{x}$ of the function $f : \mathbb{R}^n \setminus \{\vec{0}\} \to \mathbb{R}$ given by**

$$f(\vec{x}) \doteq \frac{\vec{x}^\top A\vec{x}}{\vec{x}^\top \vec{x}}. \tag{14}$$

Show your work.

HINT: Recall the quotient rule for finding the gradient of $h(\vec{x}) = \frac{n(\vec{x})}{d(\vec{x})}$ where n and d are scalar-valued functions:

$$\nabla h(\vec{x}) = \frac{d(\vec{x})\nabla n(\vec{x}) - n(\vec{x})\nabla d(\vec{x})}{(d(\vec{x}))^2}. \tag{15}$$

Solution: Define $n(\vec{x}) = \vec{x}^\top A\vec{x}$ and $d(\vec{x}) = \vec{x}^\top \vec{x}$. The gradients of these functions are

$$\nabla n(\vec{x}) = (A + A^\top)\vec{x} = 2A\vec{x} \tag{16}$$

$$\nabla d(\vec{x}) = 2\vec{x}. \tag{17}$$

Then $f(\vec{x}) = \frac{n(\vec{x})}{d(\vec{x})}$, so we have

$$\begin{aligned}
\nabla f(\vec{x}) &= \frac{d(\vec{x})\nabla n(\vec{x}) - n(\vec{x})\nabla d(\vec{x})}{(d(\vec{x}))^2} \\
&= \frac{[\vec{x}^\top \vec{x}][2A\vec{x}] - [\vec{x}^\top A\vec{x}][2\vec{x}]}{(\vec{x}^\top \vec{x})^2} \\
&= 2\frac{A\vec{x}\vec{x}^\top \vec{x} - \vec{x}\vec{x}^\top A\vec{x}}{(\vec{x}^\top \vec{x})^2} \\
&= \frac{2}{\vec{x}^\top \vec{x}}\left(A - \frac{\vec{x}^\top A\vec{x}}{\vec{x}^\top \vec{x}}I\right)\vec{x}.
\end{aligned}$$

(b) (6 pts) Let $\vec{u} \in \mathbb{R}^n$. **Compute the Jacobian with respect to $\vec{x}$ of the function $\vec{g} : \mathbb{R}^n \to \mathbb{R}^n$ given by**

$$\vec{g}(\vec{x}) \doteq \vec{x}(\vec{x}^\top \vec{u}). \tag{18}$$

Show your work.

Solution: Define $h(\vec{x}) = \vec{u}^\top \vec{x}$. Then $\frac{\partial h}{\partial x_i}(\vec{x}) = u_i$. Also, define $g_i(\vec{x}) = h(\vec{x})x_i$ and we can compute the partial derivatives as:

$$\frac{\partial g_i}{\partial x_i}(\vec{x}) = h(\vec{x}) + x_i\frac{\partial h}{\partial x_i}(\vec{x}) = \vec{u}^\top \vec{x} + u_i x_i \tag{19}$$

$$\frac{\partial g_i}{\partial x_j}(\vec{x}) = x_i\frac{\partial h}{\partial x_j}(\vec{x}) = x_i u_j. \tag{20}$$

If we stack these partial derivatives in a Jacobian matrix it follows that:

$$D\vec{g}(\vec{x}) = \vec{x}\vec{u}^\top + (\vec{u}^\top \vec{x})I. \tag{21}$$

8. Gradient Descent (18 pts)

Let $\vec{y} \in \mathbb{R}^n$ be a fixed and known vector. In this problem we will use gradient descent to solve the following problem:

$$\min_{\vec{x} \in \mathbb{R}^n} f_0(\vec{x}) \tag{22}$$

$$\text{where} \quad f_0(\vec{x}) \doteq \frac{1}{2} \|\vec{x} - \vec{y}\|_2^2 . \tag{23}$$

(a) (4 pts) **Is $f_0(\vec{x})$ a convex function?** *Justify your answer.*

Solution: We compute

$$\nabla f_0(\vec{x}) = \vec{x} - \vec{y} \qquad \text{and} \qquad \nabla^2 f_0(\vec{x}) = I. \tag{24}$$

Thus, the objective function is convex as its Hessian is I, which is PSD.

We run gradient descent on f_0 with step size $\eta > 0$ and initialization $\vec{x}_0$, obtaining the iterates $\vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots$.

(b) (5 pts) **Prove that for each $t \geq 0$ we have**

$$\vec{x}_t - \vec{y} = (1-\eta)^t (\vec{x}_0 - \vec{y}). \tag{25}$$

Solution: We have

$$\begin{aligned} \vec{x}_{t+1} &= \vec{x}_t - \eta \nabla f_0(\vec{x}_t) && (26)\\ &= \vec{x}_t - \eta(\vec{x}_t - \vec{y}) && (27)\\ &= (1-\eta)\vec{x}_t + \eta \vec{y}. && (28) \end{aligned}$$

Then subtracting $\vec{y}$ from both sides, we get

$$\begin{aligned} \vec{x}_{t+1} - \vec{y} &= (1-\eta)\vec{x}_t + \eta\vec{y} - \vec{y} && (29)\\ &= (1-\eta)\vec{x}_t - (1-\eta)\vec{y} && (30)\\ &= (1-\eta)(\vec{x}_t - \vec{y}). && (31) \end{aligned}$$

By induction we have

$$\vec{x}_t - \vec{y} = (1-\eta)^t (\vec{x}_0 - \vec{y}). \tag{32}$$

(c) (4 pts) **Determine the range of $\eta \in \mathbb{R}$ such that, for *all* initializations $\vec{x}_0$, we have $\vec{x}_1 = \vec{y}$.** *Justify your answer.*

Solution: We have

$$\vec{x}_1 - \vec{y} = (1-\eta)(\vec{x}_0 - \vec{y}). \tag{33}$$

For this to be $\vec{0}$ for all choices of $\vec{x}_0$ we must have $1 - \eta = 0$, i.e., $\eta = 1$.

(d) (5 pts) **Determine the range of $\eta \in \mathbb{R}$ such that, for *all* initializations $\vec{x}_0$ and *all* $t \geq 0$, we have that $\vec{x}_t$ is a *convex* combination of $\vec{x}_0$ and $\vec{y}$.** *Justify your answer.*

HINT: What happens if we add $\vec{y}$ to both sides of (25)?

Solution: We have

$$\begin{aligned}\vec{x}_t &= (1-\eta)^t(\vec{x}_0 - \vec{y}) + \vec{y} \\ &= (1-\eta)^t\vec{x}_0 + (1-(1-\eta)^t)\vec{y}.\end{aligned}$$

This is a convex combination if and only if $(1-\eta)^t \in [0,1]$. If t is even then $(1-\eta)^t \in [0,1]$ if and only if $1-\eta \in [-1,1]$ if and only if $\eta \in [0,2]$. If t is odd then $(1-\eta)^t \in [0,1]$ if and only if $1-\eta \in [0,1]$ if and only if $\eta \in [0,1]$. Since we want $(1-\eta)^t \in [0,1]$ for all t, this is true if and only if $\eta \in [0,1] \cap [0,2] = [0,1]$.

9. Shift Matrix (10 pts)

Let $V \in \mathbb{R}^{n\times n}$ be a square orthonormal matrix, i.e., its columns are orthogonal and have norm 1:

$$V = \begin{bmatrix} \uparrow & \uparrow & \cdots & \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_{n-1} & \vec{v}_n \\ \downarrow & \downarrow & \cdots & \downarrow & \downarrow \end{bmatrix}. \tag{34}$$

Now, we define the shifted matrix $W \in \mathbb{R}^{n\times n}$, which is composed of the columns of V shifted to the left by 1 index and padded by a zero vector:

$$W = \begin{bmatrix} \uparrow & \uparrow & \cdots & \uparrow & \uparrow \\ \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n & \vec{0} \\ \downarrow & \downarrow & \cdots & \downarrow & \downarrow \end{bmatrix}. \tag{35}$$

(a) (4 pts) **What is** $\text{rank}(V)$**? What about** $\text{rank}(W)$**?** *You do not need to justify your answers.*

Solution: V is orthogonal, so it has full column rank. Therefore $\text{rank}(V) = n$. Since $\{v_1, \ldots, v_n\}$ is a set of n linearly independent vectors, $\{v_2, \ldots, v_n\}$ is a set of $n-1$ linearly independent vectors. $\vec{0}$ is linearly dependent to all other vectors, so there are just $n-1$ linearly independent columns of W. Therefore $\text{rank}(W) = n - 1$.

(b) (6 pts) **Find a basis for the null space of** $V - W$ **and compute** $\text{rank}(V - W)$**.** *Show your work.*

HINT: Write out the definition of null space for $V - W$.

Solution: Suppose $\vec{x} \in \mathcal{N}(V - W)$. Then,

$$\vec{0} = (V - W)\vec{x} \tag{36}$$

$$\implies \vec{0} = \left[\sum_{i=1}^{n-1} x_i(\vec{v}_i - \vec{v}_{i+1})\right] + x_n\vec{v}_n \tag{37}$$

$$\implies \vec{0} = x_1\vec{v}_1 + \left[\sum_{i=1}^{n-1}(x_{i+1} - x_i)\vec{v}_{i+1}\right] \tag{38}$$

Since v_i are all linearly independent, this implies that $x_n = \ldots = x_1 = 0$, which means that the null space is trivial. By rank-nullity, this means that $\text{rank}(V - W) = n - \dim(\mathcal{N}(V - W)) = n$

10. Symmetric Matrices (10 pts)

Let $A \in \mathbb{R}^{n \times n}$ be a square matrix.

(a) (4 pts) **Prove that if A is symmetric then A^{2k} is symmetric positive semidefinite for all integers $k > 1$.**

Solution: Many different proofs. Diagonalizing A we get $A = U\Lambda U^\top$. Then $A^k = U\Lambda^{2k}U^\top$ and $\Lambda^{2k} \succeq 0$.

(b) (6 pts) **Prove that if A is symmetric then its *matrix exponential*, defined as $\mathrm{e}^A \in \mathbb{R}^{n \times n}$ given by**

$$\mathrm{e}^A = I + A + \frac{1}{2}A^2 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k \tag{39}$$

is symmetric positive definite.

HINT: The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = \mathrm{e}^x$ has the series definition

$$\mathrm{e}^x = 1 + x + \frac{1}{2}x^2 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}x^k. \tag{40}$$

Solution: Diagonalizing $A = U\Lambda U^\top$, we get

$$\mathrm{e}^A = \sum_{k=0}^{\infty} \frac{1}{k!}A^k \tag{41}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}(U\Lambda U^\top)^k \tag{42}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}U\Lambda^k U^\top \tag{43}$$

$$= U\left(\sum_{k=0}^{\infty} \frac{1}{k!}\Lambda^k\right)U^\top \tag{44}$$

$$= U\left(\sum_{k=0}^{\infty} \frac{1}{k!}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}^k\right)U^\top \tag{45}$$

$$= U\begin{bmatrix} \sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!} & & \\ & \ddots & \\ & & \sum_{k=0}^{\infty} \frac{\lambda_n^k}{k!} \end{bmatrix}U^\top \tag{46}$$

$$= U\begin{bmatrix} \mathrm{e}^{\lambda_1} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_n} \end{bmatrix}U^\top. \tag{47}$$

This is a symmetric matrix whose eigenvalues are $\mathrm{e}^{\lambda_i} > 0$, hence it is positive definite.

11. Second Principal Component (8 pts)

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with eigenvalue-eigenvector pairs given by $(\lambda_1, \vec{v}_1), \ldots, (\lambda_n, \vec{v}_n)$, where $\lambda_1 > \cdots > \lambda_n$. Consider the problem

$$\begin{aligned} p^\star = \max_{\vec{x}\in\mathbb{R}^n} \quad & \vec{x}^\top A\vec{x} \\ \text{s.t.} \quad & \|\vec{x}\|_2^2 = 1 \\ & \vec{x}^\top \vec{v}_1 = 0. \end{aligned} \tag{48}$$

Show that $p^\star = \lambda_2$**.** *Prove your answer.*

HINT: First find an $\vec{x}$ which is feasible and $\vec{x}^\top A\vec{x} = \lambda_2$. Then show that $p^\star \leq \lambda_2$.

Solution: Note that $\vec{x} = \vec{v}_2$ is feasible since $\|\vec{v}_2\| = 1$ and $\vec{v_2}^\top \vec{v}_1 = 0$ because $\vec{v}_1$ and $\vec{v}_1$ are vectors of distinct eigenspaces. Since $\vec{v}_2^\top A\vec{v}_2 = \lambda_2$, we are done if we can show that $p^\star \leq \lambda_2$.

Note that $\vec{v}_1, \ldots, \vec{v}_n$ form an orthonormal eigenbasis by the spectral theorem, which furnishes A with an orthonormal diagonalization $A = V\Lambda V^\top$, where $V = \begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{bmatrix}$ and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. For $\vec{x}$ to be feasible, we require that $\vec{x}^\top \vec{v}_1 = 0$ which implies that $\vec{x} \in \operatorname{span}(\vec{v}_1)^\perp = \operatorname{span}(\vec{v}_2, \ldots, \vec{v}_n)$. Hence, we can represent $\vec{x}$ as a linear combination:

$$\vec{x} = y_2\vec{v}_2 + \cdots + y_n\vec{v}_n = \begin{bmatrix}\vec{v}_2 & \cdots & \vec{v}_n\end{bmatrix}\begin{bmatrix}y_2\\ \vdots \\ y_n\end{bmatrix} = \begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{bmatrix}\begin{bmatrix}0\\ y_2\\ \vdots \\ y_n\end{bmatrix} = V\vec{y} \tag{49}$$

where

$$\vec{y} = \begin{bmatrix}0\\ y_2\\ \vdots \\ y_n\end{bmatrix}. \tag{50}$$

Combining everything,

$$\begin{aligned} \vec{x}^\top A\vec{x} &= \vec{x}^\top V\Lambda V^\top \vec{x} && (51)\\ &= \vec{y}^\top V^\top V\Lambda V^\top V\vec{y} && (52)\\ &= \vec{y}^\top \Lambda\vec{y} && (53)\\ &= \begin{bmatrix}0 & y_2 & \cdots & y_n\end{bmatrix}\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}\begin{bmatrix}0\\ y_2\\ \vdots \\ y_n\end{bmatrix} && (54)\\ &= \sum_{i=2}^n \lambda_i y_i^2. && (55) \end{aligned}$$

Since V is orthonormal, we have $\|\vec{y}\|_2 = \left\|V^\top \vec{x}\right\|_2 = \|\vec{x}\|_2$, so the optimization problem above can be restated as

$$p^\star = \max_{\substack{\vec{y}\in\mathbb{R}^n\\ \|\vec{y}\|_2=1\\ y_1=0}} \vec{y}^\top \Lambda\vec{y} \tag{56}$$

$$= \max_{\substack{\vec{y} \in \mathbb{R}^n \\ \|\vec{y}\|_2 = 1 \\ y_1 = 0}} \sum_{i=2}^{n} \lambda_i y_i^2 \tag{57}$$

$$\leq \lambda_2 \cdot \max_{\substack{\vec{y} \in \mathbb{R}^n \\ \|\vec{y}\|_2 = 1 \\ y_1 = 0}} \sum_{i=2}^{n} y_i^2 \tag{58}$$

$$= \lambda_2 \cdot \max_{\substack{\vec{y} \in \mathbb{R}^n \\ \|\vec{y}\|_2 = 1 \\ y_1 = 0}} \|\vec{y}\|_2^2 \tag{59}$$

$$= \lambda_2 \cdot 1 \tag{60}$$

$$= \lambda_2. \tag{61}$$

Thus, $p^\star \leq \lambda_2$ and we are done.

12. Block Ridge Regression (13 pts)

In this problem, we consider a certain generalization of ridge regression. For $d > 0$, let $A \in \mathbb{R}^{n\times(3d)}$ and $y \in \mathbb{R}^n$. Let $\vec{x}_1, \vec{x}_2, \vec{x}_3 \in \mathbb{R}^d$ be three vectors, each of dimension d. We associate regularization parameter λ_i^2 to each vector $\vec{x}_i$. We stack the $\vec{x}_i$ up to get a long $3d$-dimensional vector $\vec{x} \in \mathbb{R}^{3d}$:

$$\vec{x} = \begin{bmatrix}\vec{x}_1\\ \vec{x}_2\\ \vec{x}_3\end{bmatrix}. \tag{62}$$

With this notation, the block ridge regression problem is

$$\vec{x}_{\mathrm{BRR}} = \operatorname*{argmin}_{\vec{x}\in\mathbb{R}^{3d}} f_0(\vec{x}) \tag{63}$$

$$\text{where} \qquad f_0(\vec{x}) \doteq \|A\vec{x}-\vec{y}\|_2^2 + \sum_{i=1}^{3}\lambda_i^2\|\vec{x}_i\|_2^2 \tag{64}$$

$$= \|A\vec{x}-\vec{y}\|_2^2 + \|D\vec{x}\|_2^2 \quad \text{for} \quad D = \begin{bmatrix}\lambda_1 I_d & 0_{d\times d} & 0_{d\times d}\\ 0_{d\times d} & \lambda_2 I_d & 0_{d\times d}\\ 0_{d\times d} & 0_{d\times d} & \lambda_3 I_d\end{bmatrix}. \tag{65}$$

(a) (6 pts) **Compute $\nabla f_0(\vec{x})$.** *Show your work.*

Solution: We have that the gradients in $\vec{x}$ are

$$\nabla\|A\vec{x}-\vec{y}\|_2^2 = 2A^\top(A\vec{x}-\vec{y}) = 2A^\top Ax - 2A^\top\vec{y} \tag{66}$$

$$\nabla\|D\vec{x}\|_2^2 = 2D^\top D\vec{x} = 2D^2\vec{x}. \tag{67}$$

Thus summing them up, we get that the gradient is

$$\nabla f_0(\vec{x}) = 2(A^\top A\vec{x} - A^\top y + D\vec{x}) = 2(A^\top A + D^2)\vec{x} - 2A^\top\vec{y}. \tag{68}$$

(b) (7 pts) Recall the solution to the ridge regression problem given in class, i.e., $\vec{x}_{\mathrm{RR}} = (A^\top A + \lambda I)^{-1}A^\top\vec{y}$. **Give an expression for $\vec{x}_{\mathrm{BRR}}$ that is similar in structure to the expression for $\vec{x}_{\mathrm{RR}}$.** *Justify your answer.*

Solution: The Hessian of f_0 is

$$\nabla^2 f_0(\vec{x}) = 2A^\top A + 2D^2. \tag{69}$$

We claim that it is PD. Indeed, for $\vec{w} \neq \vec{0}$, we have

$$\vec{w}^\top[\nabla^2 f_0(\vec{x})]\vec{w} = \vec{w}^\top(2A^\top A + 2D^2)\vec{w} \tag{70}$$

$$= 2\vec{w}^\top A^\top A\vec{w} + 2\vec{w}^\top D^2\vec{w} \tag{71}$$

$$= 2\underbrace{\|A\vec{w}\|_2^2}_{\geq 0} + 2\underbrace{\|D\vec{w}\|_2^2}_{>0} \tag{72}$$

$$> 0. \tag{73}$$

Thus $\nabla^2 f_0(\vec{x})$ is PD. Moreover, $A^\top A + D^2$ has all positive eigenvalues and so is invertible.

The former fact gives that f_0 is convex. Thus we can take the gradient and set it to 0 to obtain the solution. We get

$$\vec{0} = \nabla f_0(\vec{x}_{\mathrm{BRR}}) \tag{74}$$

$$= 2(A^\top A + D^2)\vec{x}_{\mathrm{BRR}} - 2A^\top \vec{y} \quad (75)$$

$$\implies (A^\top A + D^2)\vec{x}_{\mathrm{BRR}} = A^\top \vec{y} \quad (76)$$

$$\implies \vec{x}_{\mathrm{BRR}} = (A^\top A + D^2)^{-1} A^\top \vec{y}. \quad (77)$$

13. Low-Rank Matrix Completion (28 pts)

Consider a matrix $A \in \mathbb{R}^{m \times n}$. If some entries are corrupted, one principled way to identify A is to find the matrix $B \in \mathbb{R}^{m \times n}$ of minimal rank that agrees with A on all known entries. This can be formulated as an optimization problem whose objective function is $\operatorname{rank}(B)$. Because the $\operatorname{rank}(\cdot)$ function is not continuous, we use the intuition that a low-rank matrix will only have a few nonzero singular values, and instead use the sum-of-singular-values function as the objective:

$$f(B) \doteq \sum_{i=1}^{\operatorname{rank}(B)} \sigma_i\{B\} \tag{78}$$

where $\sigma_i\{B\}$ is the i^{th} largest singular value of B. In this problem we will explore some properties of f.

(a) (8 pts) **Prove that**

$$f(B) \leq \max_{\substack{C \in \mathbb{R}^{m \times n} \\ \|C\|_2 \leq 1}} \operatorname{tr}\left(C^\top B\right). \tag{79}$$

Here $\operatorname{tr}(\cdot)$ is the trace, which for a matrix $X \in \mathbb{R}^{m \times n}$ with entries X_{ij} is $\operatorname{tr}(X) = \sum_{i=1}^{\min\{m,n\}} X_{ii}$.

HINT: Expand B into its SVD. Try to find a $D \in \mathbb{R}^{m \times n}$ such that $\|D\|_2 = 1$ and $\operatorname{tr}\left(D^\top B\right) = f(B)$.

HINT: You may use the cyclic property of traces without proof. If XYZ and ZXY are valid matrix products then $\operatorname{tr}(XYZ) = \operatorname{tr}(ZXY)$.

Solution: Let $r \doteq \operatorname{rank}(B)$. Let $B = U_r \Sigma_r V_r^\top$ be the compact SVD of B. Let $D = U_r V_r^\top$. Then $\|D\|_2 = 1$, so

$$\begin{aligned}
\max_{\substack{C \in \mathbb{R}^{m \times n} \\ \|C\|_2 \leq 1}} \operatorname{tr}\left(C^\top B\right) &\geq \operatorname{tr}\left(D^\top B\right) && (80)\\
&= \operatorname{tr}\left(V_r U_r^\top U_r \Sigma_r V_r^\top\right) && (81)\\
&= \operatorname{tr}\left(V_r \Sigma_r V_r^\top\right) && (82)\\
&= \operatorname{tr}\left(V_r^\top V_r \Sigma_r\right) && (83)\\
&= \operatorname{tr}(\Sigma_r) && (84)\\
&= \sum_{i=1}^{r} \sigma_i\{B\} && (85)\\
&= f(B). && (86)
\end{aligned}$$

(b) (9 pts) **Prove that**

$$f(B) \geq \max_{\substack{C \in \mathbb{R}^{m \times n} \\ \|C\|_2 \leq 1}} \operatorname{tr}\left(C^\top B\right). \tag{87}$$

Here $\operatorname{tr}(\cdot)$ is the trace, which for a matrix $X \in \mathbb{R}^{m \times n}$ with entries X_{ij} is $\operatorname{tr}(X) = \sum_{i=1}^{\min\{m,n\}} X_{ii}$.

HINT: Let $r \doteq \operatorname{rank}(B)$ and expand B into its outer product SVD, i.e., $B = \sum_{i=1}^{r} \sigma_i\{B\} \vec{u}_i \vec{v}_i^\top$.

HINT: You may use the cyclic and linearity properties of traces without proof. If XYZ and ZXY are valid matrix products then $\operatorname{tr}(XYZ) = \operatorname{tr}(ZXY)$. Also, $\operatorname{tr}(\alpha X + \beta Y) = \alpha \operatorname{tr}(X) + \beta \operatorname{tr}(Y)$ for $\alpha, \beta \in \mathbb{R}$.

Solution: Let $r = \operatorname{rank}(B)$. Let $B = \sum_{i=1}^{r} \sigma_i\{B\}\vec{u}_i\vec{v}_i^\top$ be an outer product SVD of B. For any $C \in \mathbb{R}^{m\times n}$ such that $\|C\|_2 \leq 1$, we have

$$\begin{aligned}
\operatorname{tr}\left(C^\top B\right) &= \operatorname{tr}\left(C^\top \left(\sum_{i=1}^{r} \sigma_i\{B\}\vec{u}_i\vec{v}_i^\top\right)\right) && (88)\\
&= \operatorname{tr}\left(\sum_{i=1}^{r} \sigma_i\{B\}C^\top\vec{u}_i\vec{v}_i^\top\right) && (89)\\
&= \sum_{i=1}^{r} \sigma_i\{B\}\operatorname{tr}\left(C^\top\vec{u}_i\vec{v}_i^\top\right) && (90)\\
&= \sum_{i=1}^{r} \sigma_i\{B\}\operatorname{tr}\left(\vec{v}_i^\top C^\top\vec{u}_i\right) && (91)\\
&= \sum_{i=1}^{r} \sigma_i\{B\}(\vec{v}_i^\top C^\top\vec{u}_i) && (92)\\
&\leq \sum_{i=1}^{r} \sigma_i\{B\}\left\|C\vec{v}_i\right\|_2\left\|\vec{u}_i\right\|_2 && (93)\\
&\leq \sum_{i=1}^{r} \sigma_i\{B\}\underbrace{\|C\|_2}_{\leq 1}\underbrace{\|\vec{v}_i\|_2}_{=1}\underbrace{\|\vec{u}_i\|_2}_{=1} && (94)\\
&\leq \sum_{i=1}^{r} \sigma_i\{B\} && (95)\\
&= f(B). && (96)
\end{aligned}$$

This holds for all C such that $\|C\|_2 \leq 1$, so taking the max over C gets

$$f(B) \geq \max_{\substack{C\in\mathbb{R}^{m\times n}\\ \|C\|_2\leq 1}} \operatorname{tr}\left(C^\top B\right) \tag{97}$$

as desired.

From parts (a) and (b) together, we can conclude that

$$f(B) = \max_{\substack{C\in\mathbb{R}^{m\times n}\\ \|C\|_2\leq 1}} \operatorname{tr}\left(C^\top B\right). \tag{98}$$

Here $\operatorname{tr}(\cdot)$ is the trace, which for a matrix $X \in \mathbb{R}^{m\times n}$ with entries X_{ij} is $\operatorname{tr}(X) = \sum_{i=1}^{\min\{m,n\}} X_{ii}$.

(c) (6 pts) **Show that for all $B_1, B_2 \in \mathbb{R}^{m\times n}$ we have**

$$f(B_1 + B_2) \leq f(B_1) + f(B_2) \tag{99}$$

i.e., the function f satisfies the triangle inequality.

HINT: Use the characterization of f given by (98). Also, you may use the linearity property of traces without proof, i.e., $\operatorname{tr}(\alpha X + \beta Y) = \alpha\operatorname{tr}(X) + \beta\operatorname{tr}(Y)$ *for* $\alpha, \beta \in \mathbb{R}$.

Solution: We have

$$f(B_1 + B_2) = \max_{\substack{C\in\mathbb{R}^{m\times n}\\ \|C\|_2\leq 1}} \operatorname{tr}\left(C^\top(B_1 + B_2)\right) \tag{100}$$

$$= \max_{\substack{C \in \mathbb{R}^{m \times n} \\ \|C\|_2 \leq 1}} \operatorname{tr}\left(C^\top B_1 + C^\top B_2\right) \tag{101}$$

$$= \max_{\substack{C \in \mathbb{R}^{m \times n} \\ \|C\|_2 \leq 1}} \left\{\operatorname{tr}\left(C^\top B_1\right) + \operatorname{tr}\left(C^\top B_2\right)\right\} \tag{102}$$

$$\leq \max_{\substack{C \in \mathbb{R}^{m \times n} \\ \|C\|_2 \leq 1}} \operatorname{tr}\left(C^\top B_1\right) + \max_{\substack{C \in \mathbb{R}^{m \times n} \\ \|C\|_2 \leq 1}} \operatorname{tr}\left(C^\top B_2\right) \tag{103}$$

$$= f(B_1) + f(B_2). \tag{104}$$

Indeed, f is a norm; note that f is called the *nuclear norm*, and it is quite relevant in signal processing. The reason that the nuclear norm is a good approximation to the rank function is because the nuclear norm is the so-called *convex envelope* of the rank function – the largest convex function which is a lower bound to the rank function. Intuitively a low-rank matrix should have only a few nonzero singular values, and the nuclear norm promotes this behavior. This has the same geometric picture as LASSO – using the ℓ^1 norm objective function to enforce that the solution has only a few nonzero entries – which we will explore later in the class.

(d) (5 pts) **Is f a convex function? If yes, prove it. If no, justify your answer using an example.**

HINT: You can use without proof the fact that $f(\alpha B) = |\alpha| f(B)$ for all $\alpha \in \mathbb{R}$ and $B \in \mathbb{R}^{m \times n}$.

Solution: Yes, f is convex; the triangle inequality and the hint give

$$f(\theta B_1 + (1-\theta)B_2) \leq f(\theta B_1) + f((1-\theta)B_2) = \theta f(B_1) + (1-\theta) f(B_2) \tag{105}$$

for $B_1, B_2 \in \Omega$ and $\theta \in [0, 1]$.