Final

1. **Honor Code (0 pts)**

   Please copy the following statement in the space provided below and sign your name.

   As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.

   If you do not copy the honor code and sign your name, you will get a 0 on the exam.

   Solution:

2. **SID (2 pts)**

   When the exam starts, write your SID at the top of every page. *No extra time will be given for this task.*

3. **Favorites (2 pts)**

   (a) (1 pts) What is your favorite song or piece of music?

   Solution: Any answer is fine.

   (b) (1 pts) What is your favorite hobby or pastime?

   Solution: Any answer is fine.

4. Singular Values (6 pts)

Suppose that $A \in \mathbb{R}^{4\times 3}$ has singular values 0, 1, $\sqrt{5}$, and 3. Let $B = \begin{bmatrix} A & 2I_4 & 3A \end{bmatrix} \in \mathbb{R}^{4\times 10}$, where $I_4 \in \mathbb{R}^{4\times 4}$ is the 4×4 identity matrix. **What are the nonzero singular values of B?** *Show your work and justify your answer(s).*

HINT: Consider the matrix $BB^\top \in \mathbb{R}^{4\times 4}$.

Solution: To find the singular values of B, we consider $BB^\top \in \mathbb{R}^{4\times 4}$. Note that we consider this matrix rather than $B^\top B \in \mathbb{R}^{10\times 10}$ because the former is smaller, and we get the following simplification:

$$BB^\top = \begin{bmatrix} A & 2I & 3A \end{bmatrix} \begin{bmatrix} A^\top \\ 2I \\ 3A^\top \end{bmatrix} = AA^\top + 4I + 9AA^\top = 10AA^\top + 4I. \tag{1}$$

By the shift and scale properties of eigenvalues, the eigenvalues of $BB^\top$ are $4+10\times$ the eigenvalues of $AA^\top$. Since the eigenvalues of $AA^\top$ are the squared singular values of A, we know that the eigenvalues of $AA^\top$ are 0, 1, 5, and 9. Thus the eigenvalues of $BB^\top$ are 4, 14, 54, and 94. Thus the nonzero singular values of B are 2, $\sqrt{14}$, $\sqrt{54}$, and $\sqrt{94}$.

5. Hyperplanes (6 pts)

(a) (2 pts) Let $\vec{c}, \vec{x}_0 \in \mathbb{R}^n$, and let $\mathcal{H} \doteq \{\vec{x} \in \mathbb{R}^n : \vec{c}^\top(\vec{x} - \vec{x}_0) = 0\}$ be a hyperplane. **Describe the set of all vectors normal to** $\mathcal{H}$. *You do not need to show your work for this subpart.*

Solution: Since $\mathcal{H} = \{\vec{c}^\top \vec{x} = \vec{c}^\top \vec{x}_0\}$, the vectors normal to this hyperplane are the vectors parallel to $\vec{c}$, i.e., $\mathrm{span}(\vec{c})$.

(b) (4 pts) Let $\vec{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{R}^2$, and define the hyperplane $\mathcal{H} = \{\vec{x} \in \mathbb{R}^2 : \vec{c}^\top \vec{x} = 0\}$. Let $\vec{y} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \in \mathbb{R}^2$. **Compute the minimum distance from** $\vec{y}$ **to** $\mathcal{H}$**.** *Show your work and justify your answer(s).*

Solution: We know the projection of $\vec{y}$ onto the hyperplane, which we will denote as $\vec{p}$, must take the following form:

$$\vec{p} = \vec{y} + \alpha\vec{c}. \tag{2}$$

for some scalar $\alpha \in \mathbb{R}$. Since $\vec{p}$ lives on the hyperplane, we also know:

$$\vec{c}^\top \vec{p} = 0. \tag{3}$$

Plugging the definition of $\vec{p}$ from equation (2) into equation (3), we get:

$$\vec{c}^\top \vec{y} + \alpha \vec{c}^\top \vec{c} = 0. \tag{4}$$

Solving for α, we get:

$$\alpha = -\frac{\vec{c}^\top \vec{y}}{\|\vec{c}\|_2^2} = -\frac{2}{2} = -1. \tag{5}$$

Plugging in this value for α, we can recover $\vec{p}$:

$$\vec{p} = \vec{y} + \alpha\vec{c} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \tag{6}$$

Finally, we compute the distance between the points $\vec{y}$ and $\vec{p}$:

$$\|\vec{y} - \vec{p}\|_2 = |\alpha| \|\vec{c}\|_2 = \sqrt{2}. \tag{7}$$

6. Shadow Prices (13 pts)

You are opening a new boba tea shop, MooncoW Boba. You have some ingredients in stock, and are selling two different boba teas. The products and ingredient quantities for each tea, along with your stock, are listed below.

	Tea (Cups)	Milk (Cups)	Boba Pearls (Scoops)	Price ($)
Mango Tea, 1 Serving	3	0	1	5
Matcha Latte, 1 Serving	1	2	1	12
MooncoW Stock	300	200	150	N/A

You want to maximize the revenue of selling x servings of Mango Tea and y servings of Matcha Latte.

(a) (7 pts) The revenue maximization problem can be expressed as the following linear program (LP) $\mathcal{P}_0$:

$$\mathcal{P}_0: \qquad p^\star = \max_{x,y\in\mathbb{R}} \quad 5x + 12y \tag{8}$$

$$\text{s.t.} \quad 3x + y \leq 300, \tag{9}$$

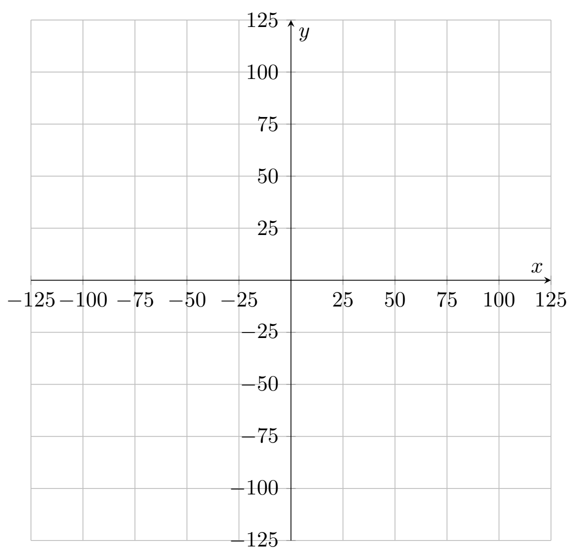
$$2y \leq 200, \tag{10}$$

$$x + y \leq 150, \tag{11}$$

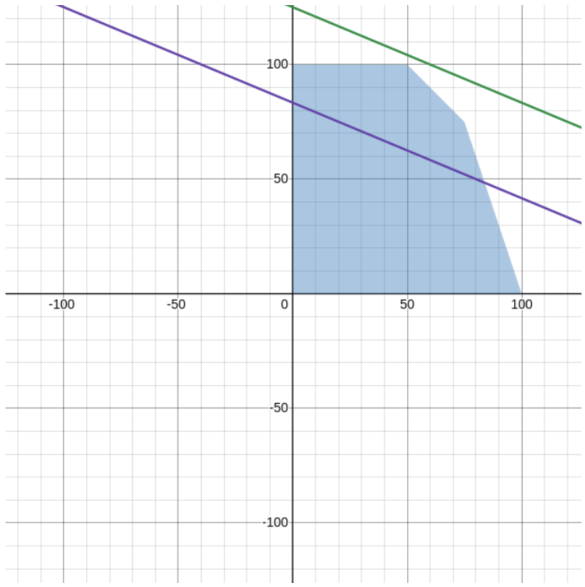
$$x \geq 0, \tag{12}$$

$$y \geq 0. \tag{13}$$

i. **Sketch and shade the feasible region of the above optimization problem in the graph provided below.**



Solution:



ii. **Use your sketch to identify the optimal value $p^\star$ and optimal point $(x^\star, y^\star)$ of the LP $\mathcal{P}_0$.** *Justify your answer(s).*

Solution: We have $\vec{x}^\star = \begin{bmatrix} 50 \\ 100 \end{bmatrix}$, so that $p^\star = 1450$. We can arrive at this solution noticing that the optimal point of an LP will be one of its vertices (as long as the feasible region is bounded, which it is in this case). We can compute the vertices either graphically or by intersecting the equations of constraints corresponding to adjacent edges of the feasible region. This gives us the set of vertices $(0,0), (0,100), (50,100), (75,75), (100,0)$. If we apply the objective function to each of these vertices, we get the corresponding objective values of $0, 1200, 1450, 1275, and 500$ respectively. Clearly, 1450 is the largest objective value, so that is our $p^\star$, with a corresponding $\vec{x}^\star$ of $\begin{bmatrix} 50 \\ 100 \end{bmatrix}$

(b) (6 pts) In order to earn more revenue, you change prices. Now, a serving of Mango Tea has a fixed price of a dollars, and a serving of Matcha Latte has a fixed price of b dollars. The new revenue maximization LP, $\mathcal{P}_1$, is as follows:

$$\mathcal{P}_1: \quad p^\star = \max_{x,y\in\mathbb{R}} \quad ax + by \tag{14}$$

$$\text{s.t.} \quad 3x + y \le 300, \tag{9}$$

$$2y \le 200, \tag{10}$$

$$x + y \le 150, \tag{11}$$

$$x \ge 0, \tag{12}$$

$$y \ge 0. \tag{13}$$

Consider the following dual variables corresponding to constraints in $\mathcal{P}_1$ and their optimal values:

Constraint	Dual Variable (λ_i)	Optimal Value ($\lambda_i^\star$)
$3x + y \le 300$	λ_1	10
$2y \le 200$	λ_2	0
$x + y \le 150$	λ_3	0
$x \ge 0$	λ_4	0
$y \ge 0$	λ_5	5

i. **Based on the table above, list the constraints of $\mathcal{P}_1$ that must be active at optimum.** *Justify your answer(s).*

Solution: The constraints $3x + y \leq 300$ and $y \geq 0$ are active constraints. This is because LPs are convex and satsfy Slater's condition for strong duality, so all points at optimum must satisfy the KKT conditions. So, an optimal point must satisfythe complementary slackness condition. So, at optimum, the constraints corresponding to strictly positive dual variables must be active.

ii. **Find, in terms of a and b, the optimal value $p^\star$ and optimal point $(x^\star, y^\star)$ of $\mathcal{P}_1$.** *Justify your answer(s).*

Solution: We can intersect the boundaries of the two active constraints. Since $y = 0$, we know that $3x = 300$, so $x = 100$. Thus, $\vec{x}^\star = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$ and $p^\star = 100a$.

7. Dual of QP (14 pts)

Let $Q \in \mathbb{R}^{n\times n}$ be a *symmetric positive definite matrix*, let $A \in \mathbb{R}^{n\times n}$ be an *invertible matrix*, and let $\vec{b} \in \mathbb{R}^n$. Consider the following quadratic program (QP):

$$p^\star = \min_{\vec{x}\in\mathbb{R}^n} \quad \frac{1}{2}\vec{x}^\top Q\vec{x} \quad \text{s.t.} \quad A\vec{x} \le \vec{b}. \tag{15}$$

(a) (10 pts) **Prove that the dual problem of (15) is given by:**

$$d^\star = \max_{\vec{\lambda}\in\mathbb{R}^n} \quad -\frac{1}{2}\vec{\lambda}^\top AQ^{-1}A^\top\vec{\lambda} - \vec{\lambda}^\top\vec{b} \quad \text{s.t.} \quad \vec{\lambda} \ge \vec{0}. \tag{16}$$

Show your work and justify your answer(s).

Solution: The Lagrangian of (15) is

$$L(\vec{x},\vec{\lambda}) = \frac{1}{2}\vec{x}^\top Q\vec{x} + \vec{\lambda}^\top(A\vec{x} - \vec{b}) \tag{17}$$

$$= \frac{1}{2}\vec{x}^\top Q\vec{x} + \vec{x}^\top A^\top\vec{\lambda} - \vec{\lambda}^\top\vec{b}. \tag{18}$$

Note that this Lagrangian is a convex quadratic function in $\vec{x}$, and therefore can be optimized by setting its gradient to $\vec{0}$. In any case, the dual function is

$$g(\vec{\lambda}) = \min_{\vec{x}\in\mathbb{R}^n} L(\vec{x},\vec{\lambda}) \tag{19}$$

$$= \min_{\vec{x}\in\mathbb{R}^n}\left[\frac{1}{2}\vec{x}^\top Q\vec{x} + \vec{x}^\top A^\top\vec{\lambda} - \vec{\lambda}^\top\vec{b}\right]. \tag{20}$$

To minimize the term inside the brackets, we compute the gradient and set it to $\vec{0}$, obtaining

$$\vec{0} = \nabla_{\vec{x}} L(\vec{x}^\star(\vec{\lambda}),\vec{\lambda}) \tag{21}$$

$$= Q\vec{x}^\star(\vec{\lambda}) + A^\top\vec{\lambda} \tag{22}$$

$$\implies \vec{x}^\star(\vec{\lambda}) = -Q^{-1}A^\top\vec{\lambda}. \tag{23}$$

The dual function is thus

$$g(\vec{\lambda}) = L(\vec{x}^\star(\vec{\lambda}),\vec{\lambda}) \tag{24}$$

$$= \frac{1}{2}(-Q^{-1}A^\top\vec{\lambda})^\top Q(-Q^{-1}A^\top\vec{\lambda}) + (-Q^{-1}A^\top\vec{\lambda})^\top A^\top\vec{\lambda} - \vec{\lambda}^\top\vec{b} \tag{25}$$

$$= \frac{1}{2}\vec{\lambda}^\top AQ^{-1}QQ^{-1}A^\top\vec{\lambda} - \vec{\lambda}^\top AQ^{-1}A^\top\vec{\lambda} - \vec{\lambda}^\top\vec{b} \tag{26}$$

$$= -\frac{1}{2}\vec{\lambda}^\top AQ^{-1}A^\top\vec{\lambda} - \vec{\lambda}^\top\vec{b}. \tag{27}$$

This is the correct dual function, and thus the dual problem which maximizes g subject to $\vec{\lambda} \ge \vec{0}$ gives (16).

(b) (4 pts) If we take the *dual of the dual* of (15), we obtain the following problem (*no need to prove this*):

$$q^\star = \max_{\vec{\mu}\in\mathbb{R}^n} \quad -\frac{1}{2}(\vec{\mu} - \vec{b})^\top(A^{-1})^\top QA^{-1}(\vec{\mu} - \vec{b}) \quad \text{s.t.} \quad \vec{\mu} \ge \vec{0}. \tag{28}$$

Prove that the maximization problem (28) is equivalent to the original quadratic program (15).

HINT: Use the fact that A is invertible to utilize the substitution $\vec{\mu} = \vec{b} - A\vec{x}$.

Solution: First, note that (28) is equivalent to the following minimization problem, which follows by negating the objective function and using the identity $-\max(-f) = \min f$:

$$\begin{aligned} \min_{\vec{\mu}\in\mathbb{R}^n} \quad & \frac{1}{2}(\vec{\mu} - \vec{b})^\top (A^{-1})^\top Q A^{-1}(\vec{\mu} - \vec{b}) \\ \text{s.t.} \quad & \vec{\mu} \geq \vec{0}. \end{aligned} \tag{29}$$

Now, following the hint, we use the substitution $\vec{\mu} = \vec{b} - A\vec{x}$, and so $\vec{\mu} - \vec{b} = A\vec{x}$. This is an invertible change of variables since A is invertible, and so it must be exactly equivalent to solve the problem over $\vec{x}$ instead of over $\vec{\mu}$. This gives

$$\begin{aligned} \min_{\vec{x}\in\mathbb{R}^n} \quad & \frac{1}{2}(A\vec{x})^\top (A^{-1})^\top Q A^{-1}(A\vec{x}) \\ \text{s.t.} \quad & \vec{b} - A\vec{x} \geq \vec{0}. \end{aligned} \tag{30}$$

Simplifying the objective gives

$$\frac{1}{2}(A\vec{x})^\top (A^{-1})^\top Q A^{-1}(A\vec{x}) = \frac{1}{2}\vec{x}^\top A^\top (A^{-1})^\top Q A^{-1} A\vec{x} = \frac{1}{2}\vec{x}^\top Q\vec{x} \tag{31}$$

and the constraint gives

$$\vec{b} - A\vec{x} \geq \vec{0} \iff \vec{b} \geq A\vec{x}, \tag{32}$$

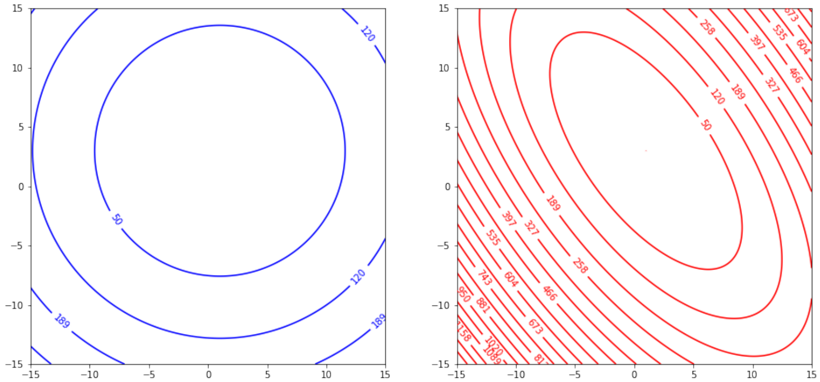
which are the objective and constraints for (15).

8. Quadratics and Gradient Descent (17 pts)

(a) (4 pts) Let $Q, R \in \mathbb{R}^{2\times 2}$ be symmetric positive definite matrices with the same *smallest* eigenvalue, i.e., $\lambda_2\{Q\} = \lambda_2\{R\}$, and different *largest* eigenvalues, i.e., $\lambda_1\{Q\} \neq \lambda_1\{R\}$. Let $\vec{b}, \vec{c} \in \mathbb{R}^2$. Let $f, g \colon \mathbb{R}^2 \to \mathbb{R}$ be such that

$$f(\vec{x}) \doteq \frac{1}{2}\vec{x}^\top Q\vec{x} + \vec{b}^\top \vec{x}, \qquad g(\vec{x}) \doteq \frac{1}{2}\vec{x}^\top R\vec{x} + \vec{c}^\top \vec{x}, \qquad \text{for all } \vec{x} \in \mathbb{R}^2. \tag{33}$$

Consider the following plots of the level sets of f (*left*) and g (*right*).



Based on these plots, which matrix, Q or R, has the larger condition number? *Justify your answer(s).*

Solution: *Short answer, perfectly acceptable:* The condition number of a 2×2 symmetric matrix can be interpreted in terms of the level sets of the associated quadratic function. Namely, the level sets form an ellipse (as described), and a more elongated ellipse directly implies a larger condition number. Thus, R has a larger condition number than Q, as its corresponding ellipse is more elongated.

Long answer, with full mathematical proof: The condition number of a $n \times n$ symmetric matrix Q can be interpreted as the ratio of the length of the longest axis to the length of the smallest axis of the generated ellipsoid, i.e.,

$$E \doteq \{\vec{x} \in \mathbb{R}^n \mid \vec{x}^\top Q\vec{x} \leq 1\}. \tag{34}$$

This is because we can write

$$\vec{x}^\top Q\vec{x} \leq 1 \tag{35}$$

$$\iff \vec{x}^\top U\Lambda U^\top \vec{x} \leq 1 \tag{36}$$

$$\iff \sum_{i=1}^{n} \lambda_i(\vec{u}_i^\top \vec{x})^2 \leq 1. \tag{37}$$

Now taking $\vec{x} = \alpha_i \vec{u}_i$ shows that the ellipsoid axis i is aligned with $\vec{u}_i$ and has length $1/\sqrt{\lambda_i}$. Thus

$$\kappa = \frac{\lambda_1}{\lambda_n} \tag{38}$$

$$= \frac{1/(\text{length of axis } 1)^2}{1/(\text{length of axis } n)^2} \tag{39}$$

$$= \left(\frac{\text{length of axis } n}{\text{length of axis } 1}\right)^2. \tag{40}$$

The upshot is that the larger the disparity of lengths is between the largest and smallest axes, the larger the condition number. And g certainly has more elongated level sets, so R has a larger condition number.

(b) (6 pts) Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, and let $\vec{b} \in \mathbb{R}^n$. Consider the optimization problem

$$\min_{\vec{x}\in\mathbb{R}^n} f(\vec{x}) \qquad \text{where} \qquad f(\vec{x}) \doteq \frac{1}{2}\vec{x}^\top Q\vec{x} + \vec{b}^\top \vec{x}. \tag{41}$$

Let $\vec{x}^\star$ solve (41). Let $(\vec{x}_t)_{t=0}^\infty$ be gradient descent iterates, with step size $\eta > 0$, for the problem (41). **Write the update rule for $\vec{x}_{t+1}$ in terms of $\vec{x}_t$. Then, use this update rule to prove that**

$$\vec{x}_t - \vec{x}^\star = (I_n - \eta Q)^t(\vec{x}_0 - \vec{x}^\star), \qquad \textbf{for all } t \geq 0. \tag{42}$$

Here $I_n \in \mathbb{R}^{n\times n}$ is the $n \times n$ identity matrix.

HINT: It may be useful to first compute $\vec{x}^\star$ in closed form.

Solution: The gradient update rule is the following,

$$\vec{x}_{t+1} = \vec{x}_t - \eta\left(Q\vec{x}_t + \vec{b}\right). \tag{43}$$

To show (42)

$$\begin{aligned}
\vec{x}_{t+1} &= \vec{x}_t - \eta\left(Q\vec{x}_t + \vec{b}\right) && (44)\\
&= (I_n - \eta Q)\vec{x}_t - \eta\vec{b} && (45)\\
\implies \vec{x}_{t+1} - \vec{x}^\star &= (I_n - \eta Q)\vec{x}_t - \eta\vec{b} + Q^{-1}\vec{b} && (46)\\
&= (I_n - \eta Q)\vec{x}_t + (I_n - \eta Q)Q^{-1}\vec{b} && (47)\\
&= (I_n - \eta Q)(\vec{x}_t - \vec{x}^\star). && (48)
\end{aligned}$$

Iterating this relation obtains the second equality.

$$\begin{aligned}
\vec{x}_t - \vec{x}^\star &= (I_n - \eta Q)(\vec{x}_{t-1} - \vec{x}^\star) && (49)\\
&= (I_n - \eta Q)(I_n - \eta Q)(\vec{x}_{t-2} - \vec{x}^\star) && (50)\\
&= \cdots && (51)\\
&= (I_n - \eta Q)^t(\vec{x}_0 - \vec{x}^\star). && (52)
\end{aligned}$$

(c) (4 pts) We now analyze the convergence of gradient descent in an eigenvector basis. Let $Q = U\Lambda U^\top$, where U is orthonormal and Λ is diagonal with entries $\lambda_1 \geq \cdots \geq \lambda_n$. Let $\vec{z}_t = U^\top\vec{x}_t$ and $\vec{z}^\star = U^\top\vec{x}^\star$. **Using (42), prove that**

$$(\vec{z}_t - \vec{z}^\star)_i = (1 - \eta\lambda_i)^t(\vec{z}_0 - \vec{z}^\star)_i, \qquad \textbf{for all } t \geq 0 \quad \textbf{and} \quad i \in \{1, \ldots, n\}, \tag{53}$$

where $(\vec{z})_i$ is the i^{th} entry of $\vec{z}$ for any vector $\vec{z}$.

Solution: We know $\vec{x}_t - \vec{x}^\star = U\vec{z}_t - U\vec{z}^\star$. From (42), we have

$$\begin{aligned}
\vec{z}_t - \vec{z}^\star &= U^\top(I_n - \eta U\Lambda U^\top)^t U(\vec{z}_0 - \vec{z}^\star) && (54)\\
&= U^\top(U(I_n - \eta\Lambda)U^\top)^t U(\vec{z}_0 - \vec{z}^\star) && (55)\\
&= (I_n - \eta\Lambda)^t(\vec{z}_0 - \vec{z}^\star). && (56)
\end{aligned}$$

Now, because $I_n - \eta\Lambda$ is a diagonal matrix with i^{th} entry $1 - \eta\lambda_i$, we have

$$\begin{aligned}
\vec{z}_t - \vec{z}^\star &= (I_n - \eta\Lambda)^t(\vec{z}_0 - \vec{z}^\star) && (57)\\
\implies (\vec{z}_t - \vec{z}^\star)_i &= (I_n - \eta\Lambda)^t_{ii}(\vec{z}_0 - \vec{z}^\star)_i && (58)\\
\implies (\vec{z}_t - \vec{z}^\star)_i &= (1 - \eta\lambda_i)^t(\vec{z}_0 - \vec{z}^\star)_i && (59)
\end{aligned}$$

(d) (3 pts) Define the interval $(\alpha, \beta) \subseteq \mathbb{R}$ such that $\lim_{t\to\infty} \vec{z}_t = \vec{z}^\star$ for any choice of initialization $\vec{z}_0$ if and only if the step size $\eta \in (\alpha, \beta)$. **Using (53), identify α and β.** *Show your work and justify your answer(s).*

Solution: Gradient descent converges for all choices of initialization if and only if $|1 - \eta\lambda_i| < 1$ for all $i \in \{1, \ldots, n\}$. Therefore,

$$-1 < 1 - \eta\lambda_i < 1, \quad \text{for all } i \in \{1, \ldots, n\} \tag{60}$$

$$\iff 0 < \eta\lambda_i < 2, \quad \text{for all } i \in \{1, \ldots, n\} \tag{61}$$

$$\iff 0 < \lambda_i < \frac{2}{\eta}, \quad \text{for all } i \in \{1, \ldots, n\} \tag{62}$$

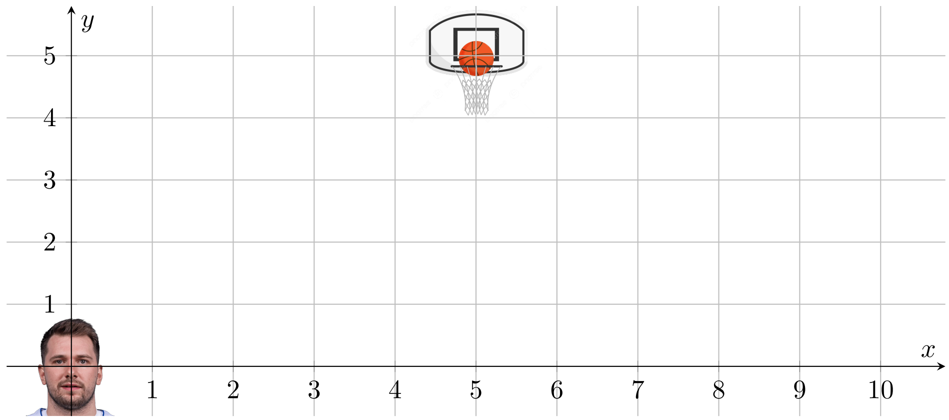
$$\iff 0 < \lambda_n \leq \lambda_1 < \frac{2}{\eta} \tag{63}$$

$$\iff 0 < \eta < \frac{2}{\lambda_1}. \tag{64}$$

Thus $\alpha = 0$ and $\beta = \frac{2}{\lambda_1}$.

9. Luka Navigates a Grid (10 pts)

Luka is playing basketball, and wants to get to the basket for a dunk. He is currently at position $\vec{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and will take 10 steps through positions $\vec{v}_1, \ldots, \vec{v}_9$ to get to position $\vec{v}_{10} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$, which is where the basket is.



(a) (4 pts) Luka has two defenders, located at points $\vec{x}, \vec{y} \in \mathbb{R}^2$, and he wants to take a path such that:

- The path maximizes the smallest distance from any location on the path, i.e., the $\vec{v}_t$, to any defender, i.e., $\vec{x}$ or $\vec{y}$.
- The path does not go out of bounds, i.e., outside the rectangle with corners $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 10 \\ 5 \end{bmatrix}$.

Formulate the optimization problem to find Luka's steps by filling in choices for (1), (2), and (3) below. *You do not need to show your work for this subpart.*

NOTE: Your problem doesn't have to be written in standard form, nor does it have to be convex.

$$\max_{\substack{q \in \mathbb{R} \\ \vec{v}_0, \vec{v}_1 \ldots \vec{v}_9, \vec{v}_{10} \in \mathbb{R}^2}} \quad q \tag{65}$$

$$\text{s.t.} \quad q \leq \|\vec{x} - \vec{v}_t\|_2, \quad \text{for all } t \in \{0, \ldots, 10\}, \tag{66}$$

$$q \leq \| \boxed{\quad (1) \quad} \|_2, \quad \text{for all } t \in \{0, \ldots, 10\}, \tag{67}$$

$$\boxed{\quad (2) \quad} \leq \vec{v}_t, \quad \text{for all } t \in \{0, \ldots, 10\}, \tag{68}$$

$$\vec{v}_t \leq \boxed{\quad (3) \quad}, \quad \text{for all } t \in \{0, \ldots, 10\}, \tag{69}$$

$$\vec{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \vec{v}_{10} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}. \tag{70}$$

Solution: Here's a possible solution, with explanations on the side:

$$\max_{\substack{q \in \mathbb{R} \\ \vec{v}_0 \ldots \vec{v}_{10} \in \mathbb{R}^2}} \quad q \quad \text{(slack variable for min. distance to obstacles)} \tag{71}$$

$$\text{s.t.} \quad q \leq \|\vec{x} - \vec{v}_t\|_2, \quad \text{for all } t \in \{0, \ldots, 10\}, \quad \text{(min. distance to enemy } \vec{x}\text{)} \tag{72}$$

$$q \leq \|\vec{y} - \vec{v}_t\|_2, \quad \text{for all } t \in \{0, \ldots, 10\}, \quad \text{(min. distance to enemy } \vec{y}\text{)} \tag{73}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \leq \vec{v}_t, \quad \text{for all } t \in \{0, \ldots, 10\}, \quad \text{(staying in bounds)} \tag{74}$$

$$\vec{v}_t \leq \begin{bmatrix} 10 \\ 5 \end{bmatrix}, \qquad \text{for all } t \in \{0, \dots, 10\}, \qquad \text{(staying in bounds)} \tag{75}$$

$$\vec{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \vec{v}_{10} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}. \qquad \text{(start and end)} \tag{76}$$

(b) (3 pts) Now, suppose each step Luka makes must be exactly one unit long in the ℓ^2-norm sense. **Write one or more constraints that capture this condition.** *You do not need to show your work for this subpart.*

NOTE: Your constraint(s) do not have to be convex.

Solution: The $\vec{v}_t$ are the positions of Luka at time t, so the step sizes are $\|\vec{v}_{t+1} - \vec{v}_t\|_2$. The constraint we should add is thus

$$\|\vec{v}_{t+1} - \vec{v}_t\|_2 = 1, \qquad \text{for all } t \in \{0, \dots, 9\}. \tag{77}$$

(c) (3 pts) Suppose $p_{\text{a}}^\star$ is the optimal value of our problem from part 9(a). Then, suppose $p_{\text{b}}^\star$ is the optimal value of this problem after we add our constraint from part 9(b). **Which of the following is guaranteed to be true?**

(A) $p_{\text{a}}^\star = p_{\text{b}}^\star$;

(B) $p_{\text{a}}^\star \leq p_{\text{b}}^\star$;

(C) $p_{\text{a}}^\star \geq p_{\text{b}}^\star$;

(D) None of the above.

Justify your answer.

Solution: We add additional constraints which can only shrink the feasible set of the optimization problem. These are maximization problems, so this means that the optimal value can only decrease. Therefore, the correct answer is (C), i.e., $p_{\text{a}}^\star \geq p_{\text{b}}^\star$.

10. Bandwidth Allocation (20 pts)

(a) (6 pts) **Consider the functions f_1 and f_2 defined below. Is f_1 convex? Is f_2 convex? For each function, if it is convex, prove it; if it is not, explain why.** *Justify your answer(s).*

i. $f_1 \colon S_1 \to \mathbb{R}$, where $S_1 \doteq \{x \in \mathbb{R} \colon x > 0\}$, and $f_1(x) \doteq 1/x$ for all $x \in S_1$.

ii. $f_2 \colon S_2 \to \mathbb{R}$, where $S_2 \doteq \{x \in \mathbb{R} \colon x \neq 0\}$, and $f_2(x) \doteq 1/x$ for all $x \in S_2$.

Solution: Note that on S_1 and S_2, the "Hessian" (second derivative) of f_i is $f''(x) \doteq \frac{\mathrm{d}^2 f_i}{\mathrm{d}x^2}(x) = 2/x^3$. Since S_1 is convex and $f_1''(x) > 0$ for $x \in S_1$, we know that f_1 is convex. However S_2 is not convex, and even still $f_2''(x) < 0$ for $x \in S_2$ such that $x < 0$. Thus f_2 is not convex for two reasons.

(b) (6 pts) In a network bandwidth allocation scenario, a total of n clients share a communication channel to send signals. The channel has an available bandwidth that has been normalized to 1. The fraction of bandwidth allocated to the i^{th} client is represented by x_i. Since the channel bandwidth is shared among n clients and every client must be allocated a non-zero fraction, $\sum_{i=1}^{n} x_i \leq 1$ and $x_i > 0$. If a client gets more bandwidth, they have better performance and their error decreases. Let the error coefficient associated with the i^{th} client be $c_i > 0$, so that the error incurred with x_i fraction of the bandwidth is c_i/x_i. The goal is to minimize the sum of incurred error among n clients. The optimization problem is formulated as follows.

$$\begin{aligned} p^\star = \min_{\vec{x} \in \mathbb{R}^n} \quad & \sum_{i=1}^{n} \frac{c_i}{x_i} \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i \leq 1, \\ & x_i > 0, \qquad \text{for all } i \in \{1, \ldots, n\}. \end{aligned} \tag{78}$$

Here, as usual, x_i denotes the i^{th} entry of $\vec{x}$. **Prove that if $\vec{x}^\star$ is an optimal solution to (78) then $\sum_{i=1}^{n} x_i^\star = 1$.**

HINT: Let $\vec{z}$ be optimal for (78) yet such that $\sum_{i=1}^{n} z_i < 1$. Let $s \doteq 1 - \sum_{i=1}^{n} z_i$ and consider $\vec{y} \doteq \vec{z} + s\vec{e}_1$, where $\vec{e}_1 \in \mathbb{R}^n$ is the first column of the identity matrix.

Solution: We claim that $\vec{y}$ is feasible; indeed,

$$\begin{aligned} \sum_{i=1}^{n} y_i &= y_1 + \sum_{i=2}^{n} y_i && (79) \\ &= z_i + s + \sum_{i=2}^{n} z_i && (80) \\ &= s + \sum_{i=1}^{n} z_i && (81) \\ &= \left(1 - \sum_{i=1}^{n} z_i\right) + \sum_{i=1}^{n} z_i && (82) \\ &= 1; && (83) \\ y_i &= \begin{cases} \underbrace{z_i}_{>0} + \underbrace{s}_{>0}, & \text{if } i = 1 \\ \underbrace{z_i}_{>0}, & \text{otherwise} \end{cases} && (84) \\ &> 0 \quad \text{for all } i. && (85) \end{aligned}$$

Now we know that $\vec{y}$ is feasible, so it remains to check the objective value. Since $s > 0$ we have

$$\sum_{i=1}^{n} \frac{c_i}{y_i} = \frac{c_1}{y_1} + \sum_{i=2}^{n} \frac{c_i}{y_i} \tag{86}$$

$$= \frac{c_1}{z_1 + s} + \sum_{i=2}^{n} \frac{c_i}{z_i} \tag{87}$$

$$< \frac{c_1}{z_i} + \sum_{i=2}^{n} \frac{c_i}{z_i} \tag{88}$$

$$= \sum_{i=1}^{n} \frac{c_i}{z_i}. \tag{89}$$

Since $\vec{y}$ is a feasible point whose objective value is strictly better than $\vec{z}$, we know that $\vec{z}$ cannot be optimal, a contradiction with the definition of $\vec{z}$.

(c) (8 pts) Now consider the following optimization problem (again with $c_1, \ldots, c_n > 0$):

$$\begin{aligned} p^\star = \min_{\vec{x} \in \mathbb{R}^n} \quad & \sum_{i=1}^{n} \frac{c_i}{x_i} \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1, \\ & x_i > 0, \qquad \text{for all } i \in \{1, \ldots, n\}. \end{aligned} \tag{90}$$

Again, as usual, x_i denotes the i^{th} entry of $\vec{x}$. In part 10(b), we proved that the earlier problem (78) is a relaxation of (90). We now solve the problem (90).

i. **First, state the Cauchy-Schwarz inequality and its equality condition.**

ii. **Next, for any $\vec{x}$ feasible for (90), find vectors $\vec{u}, \vec{w}$ such that $\|\vec{u}\|_2^2 = \sum_{i=1}^n c_i/x_i$ and $\|\vec{w}\|_2^2 = \sum_{i=1}^n x_i = 1$, and apply Cauchy-Schwarz to get a lower bound on $\sum_{i=1}^n c_i/x_i$.**

iii. **Finally, find the optimal $\vec{x}^\star$ which achieves this lower bound via the equality condition of Cauchy-Schwarz.**

Solution: The Cauchy-Schwarz inequality says that for any $\vec{u}, \vec{w} \in \mathbb{R}^n$, we have

$$(\vec{u}^\top \vec{w})^2 \leq \|\vec{u}\|_2^2 \|\vec{w}\|_2^2. \tag{91}$$

Equality holds when $\vec{u} = \alpha \vec{v}$ for some $\alpha \in \mathbb{R}$.

As per the hint, we find $\vec{u}, \vec{w}$ such that $\sum_{i=1}^n u_i^2 = \|\vec{u}\|_2^2 = \sum_{i=1}^n c_i/x_i$ and $\sum_{i=1}^n w_i^2 = \|\vec{w}\|_2^2 = \sum_{i=1}^n x_i = 1$. This is achieved by $u_i = \sqrt{c_i/x_i}$ and $w_i = \sqrt{x_i}$. Cauchy-Schwarz provides

$$(\vec{u}^\top \vec{w})^2 \leq \|\vec{u}\|_2^2 \|\vec{w}\|_2^2 \tag{92}$$

$$= \|\vec{u}\|_2^2 \tag{93}$$

$$\left(\sum_{i=1}^{n} \sqrt{\frac{c_i}{x_i}} \cdot \sqrt{x_i} \right)^2 \leq \sum_{i=1}^{n} \frac{c_i}{x_i} \tag{94}$$

$$\left(\sum_{i=1}^{n} \sqrt{c_i} \right)^2 \leq \sum_{i=1}^{n} \frac{c_i}{x_i}, \tag{95}$$

Thus

$$\sum_{i=1}^{n} \frac{c_i}{x_i} \geq \left(\sum_{i=1}^{n} \sqrt{c_i} \right)^2 \tag{96}$$

with equality if and only if $\vec{u}$ and $\vec{w}$ are collinear, i.e., $\alpha\sqrt{c_i/x_i^\star} = \sqrt{x_i^\star}$ for some $\alpha \geq 0$ and all i. This yields

$$x_i^\star = \alpha\sqrt{c_i}, \tag{97}$$

which, since $\sum_{i=1}^n x_i^\star = 1$, forces $\alpha = \frac{1}{\sum_{i=1}^n \sqrt{c_i}}$. Thus in the end we have

$$x_i^\star = \frac{\sqrt{c_i}}{\sum_{j=1}^n \sqrt{c_j}}, \qquad \text{for all } i \in \{1, \ldots, n\}. \tag{98}$$

11. Maximum Entropy (17 pts)

We aim to solve the following optimization problem:

$$p^\star = \min_{\vec{x} \in \mathbb{R}^n} \quad f(\vec{x}) \qquad \text{where} \qquad f(\vec{x}) \doteq \sum_{i=1}^{n} x_i \log(x_i) \tag{99}$$

$$\text{s.t.} \quad x_i \geq 0, \qquad \text{for all } i \in \{1, \ldots, n\}, \tag{100}$$

$$\sum_{i=1}^{n} x_i = 1, \tag{101}$$

where we adopt the convention that $0 \log(0) = 0$ so as to define f on the entire feasible set. This problem determines the probability distribution on $\{1, \ldots, n\}$ with the maximal *Shannon entropy*, although this is irrelevant to solving the problem itself.

(a) (3 pts) Let $L(\vec{x}, \vec{\lambda}, \nu)$ be the Lagrangian of the problem, where $\vec{\lambda}$ are the dual variables corresponding to the constraints in (100), and ν is the dual variable corresponding to the constraint in (101). **Prove that**

$$L(\vec{x}, \vec{\lambda}, \nu) = f(\vec{x}) - \vec{x}^\top(\vec{\lambda} - \nu\vec{1}_n) - \nu, \tag{102}$$

where $\vec{1}_n \in \mathbb{R}^n$ is the vector of ones in $\mathbb{R}^n$. *Show your work and justify your answer(s).*

Solution: We have

$$\begin{aligned} L(\vec{x}, \vec{\lambda}, \nu) &= f(\vec{x}) + \sum_{i=1}^{n} \lambda_i(-x_i) + \nu\left(\sum_{i=1}^{n} x_i - 1\right) & (103) \\ &= f(\vec{x}) - \vec{\lambda}^\top\vec{x} + \nu(\vec{1}_n^\top\vec{x} - 1) & (104) \\ &= f(\vec{x}) - \vec{x}^\top(\vec{\lambda} - \nu\vec{1}_n) - \nu & (105) \end{aligned}$$

(b) (4 pts) **Prove that strong duality holds for this problem.**

NOTE: You may assume (*i.e., do not need to prove*) that f is convex.

Solution: The objective has Hessian

$$\nabla^2 f(\vec{x}) = \begin{bmatrix} 1/x_1 & & \\ & \ddots & \\ & & 1/x_n \end{bmatrix}, \tag{106}$$

when all entries of $\vec{x}$ are strictly positive, and otherwise defined by right-continuity. Thus for any feasible $\vec{x}$, we have that $\nabla^2 f(\vec{x})$ is a diagonal matrix with non-negative entries, and hence positive semidefinite. Thus f is a convex function. Since the objective is a convex function, the feasible set is defined via affine constraints, and the feasible set is nonempty, (refined) Slater's condition yields that strong duality holds.

(c) (10 pts) The results of part 11(b), along with the fact that the problem is convex, mean that the KKT conditions are necessary and sufficient for optimality (*you do not need to prove this*). **State the KKT conditions and use them to solve for optimal primal and dual variables** $(\vec{x}^\star, \vec{\lambda}^\star, \nu^\star)$**.** *Show your work and justify your answer(s).*

HINT: First, prove that $x_i^\star = \mathrm{e}^{\lambda_i^\star - (1+\nu^\star)}$ *for each* $i \in \{1, \ldots, n\}$.

Solution: Since the KKT conditions are equivalent to optimality, let $\vec{x}^\star, \vec{\lambda}^\star, \nu^\star$ be optimal variables, so that they also satisfy the KKT conditions. The KKT conditions are:

- Primal feasibility: $x_i^\star \geq 0$ for all $i \in \{1, \ldots, n$, and $\vec{1}_n^\top \vec{x} = 1$.

- Dual feasibility: $\lambda_i^\star \geq 0$ for all $i \in \{1, \ldots, n\}$.
- Complementary slackness: $\lambda_i^\star x_i^\star = 0$ for all $i \in \{1, \ldots, n\}$.
- Stationarity:

$$\begin{aligned} \vec{0}_n &= \nabla_{\vec{x}} L(\vec{x}^\star, \vec{\lambda}^\star, \nu^\star) && (107) \\ &= \nabla f(\vec{x}^\star) - \vec{\lambda}^\star + \nu^\star \vec{1}_n. && (108) \end{aligned}$$

  Here $\vec{0}_n$ is the vector of all zeros in $\mathbb{R}^n$.

Simplifying further, we obtain

$$\begin{aligned} \nabla f(\vec{x}^\star) &= \nabla \left\{ \sum_{i=1}^n x_i^\star \log(x_i^\star) \right\} && (109) \\ &= \begin{bmatrix} 1 + \log(x_1^\star) \\ \vdots \\ 1 + \log(x_n^\star) \end{bmatrix} && (110) \\ &= \vec{1}_n + \begin{bmatrix} \log(x_1^\star) \\ \vdots \\ \log(x_n^\star) \end{bmatrix} && (111) \\ \implies \vec{0}_n &= \vec{1}_n + \begin{bmatrix} \log(x_1^\star) \\ \vdots \\ \log(x_n^\star) \end{bmatrix} - \vec{\lambda}^\star + \nu^\star \vec{1}_n && (112) \\ &= \begin{bmatrix} \log(x_1^\star) \\ \vdots \\ \log(x_n^\star) \end{bmatrix} - \vec{\lambda}^\star + (1 + \nu^\star)\vec{1}_n && (113) \\ \implies \begin{bmatrix} \log(x_1^\star) \\ \vdots \\ \log(x_n^\star) \end{bmatrix} &= \vec{\lambda}^\star - (1 + \nu^\star)\vec{1}_n && (114) \\ \implies \log(x_i^\star) &= \lambda_i^\star - (1 + \nu^\star) && (115) \\ \implies x_i^\star &= \mathrm{e}^{\lambda_i^\star - (1+\nu^\star)}. && (116) \end{aligned}$$

By complementary slackness, we have

$$0 = \lambda_i^\star x_i^\star = \lambda_i^\star \mathrm{e}^{\lambda_i^\star - (1+\nu^\star)}. \quad (117)$$

Since $\mathrm{e}^{\text{something}} > 0$ for any (real) value of something, we have that $\lambda_i^\star = 0$. Thus

$$\vec{\lambda}^\star = \vec{0}_n. \quad (118)$$

Also, this implies that

$$x_i^\star = \mathrm{e}^{-(1+\nu^\star)}, \quad (119)$$

or in other words,

$$\vec{x}^\star = \mathrm{e}^{-(1+\nu^\star)} \vec{1}_n. \quad (120)$$

To solve for $\nu^\star$, we use primal feasibility, namely

$$1 = \vec{1}_n^\top \vec{x}^\star = \vec{1}_n^\top (\mathrm{e}^{-(1+\nu^\star)} \vec{1}_n) = n\mathrm{e}^{-(1+\nu^\star)} \tag{121}$$

$$\implies \frac{1}{n} = \mathrm{e}^{-(1+\nu^\star)} \implies -\log(n) = -1 - \nu^\star \implies \nu^\star = \log(n) - 1. \tag{122}$$

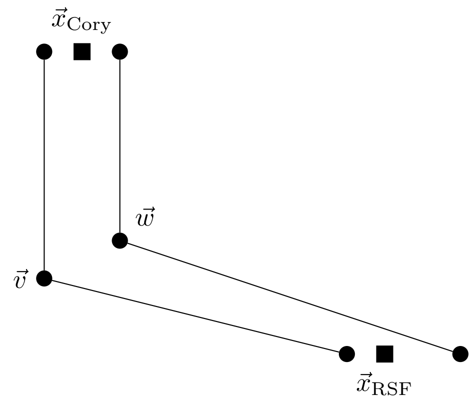
This gives

$$\vec{x}^\star = \frac{1}{n}\vec{1}_n, \qquad \vec{\lambda}^\star = \vec{0}, \qquad \vec{\nu}^\star = \log(n) - 1. \tag{123}$$

The objective corresponds to the negative *(Shannon) entropy* of a probability distribution given by $\vec{x}$, so the problem attempts to find the maximum entropy of a probability distribution on $\{1, \ldots, n\}$. This distribution turns out to be the uniform distribution, as may be intuitive if one understands Shannon entropy as a quantification of the spread of a distribution.

12. Path-Planning and SOCPs (10 pts)

Anish wants to find the best way to carry final exams from Cory Hall ($\vec{x}_{\text{Cory}}$) to RSF ($\vec{x}_{\text{RSF}}$). He identifies that the best possible route, going around the edge of campus, must stay within the following path. *NOTE*: *The diagram below is not drawn to scale.*



Here $\vec{x}_{\text{Cory}}, \vec{x}_{\text{RSF}}, \vec{v}, \vec{w}$ are points in $\mathbb{R}^2$. Anish wants to find the optimal $\vec{x}$, given by the solution to the following problem $\mathcal{P}_0$, to find the shortest path:

$$\mathcal{P}_0: \quad \min_{\substack{\vec{x}\in\mathbb{R}^2 \\ \theta\in\mathbb{R}}} \quad \|\vec{x} - \vec{x}_{\text{Cory}}\|_2 + \|\vec{x} - \vec{x}_{\text{RSF}}\|_2 \tag{124}$$

$$\text{s.t.} \quad \vec{x} = \vec{w} + \theta(\vec{v} - \vec{w}), \tag{125}$$

$$0 \le \theta \le 1. \tag{126}$$

We restrict $\vec{x}$ to be a convex combination of $\vec{v}$ and $\vec{w}$. The objective function is the total length of the path.

(a) (4 pts) **Is the problem $\mathcal{P}_0$, as written (i.e., without any reformulations), a SOCP?** *Justify your answer(s).*

Solution: No: SOCPs have linear objective and second-order cone constraints. The objective of $\mathcal{P}_0$ is not linear, nor are the constraints second-order cone constraints (although they can easily be reformulated to this form).

(b) (6 pts) We now reformulate $\mathcal{P}_0$. **Prove that the optimal $\theta^\star$ for the problem $\mathcal{P}_0$ is the same as the optimal $\theta^\star$ for the following problem $\mathcal{P}_1$:**

$$\mathcal{P}_1: \quad \min_{\theta,s,t\in\mathbb{R}} \quad s + t \tag{127}$$

$$\text{s.t.} \quad 0 \le \theta \le 1, \tag{128}$$

$$\|\theta(\vec{v} - \vec{w}) + \vec{w} - \vec{x}_{\text{Cory}}\|_2 \le s, \tag{129}$$

$$\|\theta(\vec{v} - \vec{w}) + \vec{w} - \vec{x}_{\text{RSF}}\|_2 \le t. \tag{130}$$

HINT: If you introduce inequality constraints relating to slack variables, remember to justify why they must achieve equality at the optimal solution.

Solution: Following the hint, we plug in $\vec{x}$ everywhere and eliminate it as an optimization variable. This does not change the optimal θ because $\vec{x}$ was just a placeholder variable and fixed in terms of θ. This makes the objective

$$\mathcal{P}_{1/2}: \quad \min_{\theta\in\mathbb{R}} \quad \|\theta(\vec{v} - \vec{w}) + \vec{w} - \vec{x}_{\text{Cory}}\|_2 + \|\theta(\vec{v} - \vec{w}) + \vec{w} - \vec{x}_{\text{RSF}}\|_2 \tag{131}$$

$$\text{s.t.} \quad 0 \le \theta \le 1. \tag{132}$$

Now to make the objective linear, we introduce slack variables (s, t) such that $s \geq \|\theta(\vec{v} - \vec{w}) + \vec{w} - \vec{x}_{\text{Cory}}\|_2$ and $t \geq \|\theta(\vec{v} - \vec{w}) + \vec{w} - \vec{x}_{\text{RSF}}\|_2$. These are both justified because of the epigraph reformulation:

$$\min_{\vec{x}} f(\vec{x}) = \min_{\substack{\vec{x}, t \\ t \geq f(\vec{x})}} t. \tag{133}$$

(This has been covered in the course, but is also true because the objective supports lowering t as much as possible, so $t^\star = f(\vec{x}^\star)$ at optimum, and because we also want to make $t^\star$ as small as possible we want to make $f(\vec{x}^\star)$ as small as possible, meaning that $\vec{x}^\star$ is the global minimizer of f.

This epigraph reformulation yields

$$\begin{aligned}
\mathcal{P}_1: \quad & \min_{\theta, s, t \in \mathbb{R}} \quad s + t & (134)\\
& \text{s.t.} \quad 0 \leq \theta \leq 1, & (135)\\
& \qquad \|\theta(\vec{v} - \vec{w}) + \vec{w} - \vec{x}_{\text{Cory}}\|_2 \leq s, & (136)\\
& \qquad \|\theta(\vec{v} - \vec{w}) + \vec{w} - \vec{x}_{\text{RSF}}\|_2 \leq t. & (137)
\end{aligned}$$

This is the same solution as $\mathcal{P}_1$.

13. Equivariance of Newton's Method (9 pts)

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a twice-continuously-differentiable function with invertible Hessian. Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be defined as

$$g(\vec{y}) \doteq f(A\vec{y}), \qquad \text{for all } \vec{y} \in \mathbb{R}^n. \tag{138}$$

Let $(\vec{x}_t)_{t=0}^{\infty}$ be a sequence of Newton's method iterates on f starting from $\vec{x}_0 \in \mathbb{R}^n$, and let $(\vec{y}_t)_{t=0}^{\infty}$ be a sequence of Newton's method iterates on g starting from $\vec{y}_0 = A^{-1}\vec{x}_0$. **Prove that $\vec{y}_t = A^{-1}\vec{x}_t$ for all t.**

HINT: Use induction on t, with base case $t = 0$.

Solution: We prove this by induction. The base case holds since $\vec{y}_0 = A^{-1}\vec{x}_0$ by assumption. Now by chain rule, we have

$$\begin{aligned}
\nabla g(\vec{y}) &= [Dg(\vec{y})]^\top && (139)\\
&= [(Df(A\vec{y}))(D_{\vec{y}}\{A\vec{y}\})]^\top && (140)\\
&= [(Df(A\vec{y}))A]^\top && (141)\\
&= A^\top \nabla f(A\vec{y}) && (142)\\
\nabla^2 g(\vec{y}) &= D[\nabla g](\vec{y}) && (143)\\
&= D[A^\top \nabla f(A\vec{y})] && (144)\\
&= A^\top D[\nabla f(A\vec{y})] && (145)\\
&= A^\top \nabla^2 f(A\vec{y})A. && (146)
\end{aligned}$$

Thus

$$\begin{aligned}
\vec{y}_{t+1} &= \vec{y}_t - [\nabla^2 g(\vec{y}_t)]^{-1}[\nabla g(\vec{y}_t)] && (147)\\
&= \vec{y}_t - [A^\top \nabla^2 f(A\vec{y}_t)A]^{-1}A^\top \nabla f(A\vec{y}_t) && (148)\\
&= \vec{y}_t - A^{-1}[\nabla^2 f(A\vec{y}_t)]^{-1}(A^\top)^{-1}A^\top \nabla f(A\vec{y}_t) && (149)\\
&= \vec{y}_t - A^{-1}[\nabla^2 f(A\vec{y}_t)]^{-1}\nabla f(A\vec{y}_t) && (150)\\
A\vec{y}_{t+1} &= A\vec{y}_t - [\nabla^2 f(A\vec{y}_t)]^{-1}\nabla f(A\vec{y}_t) && (151)\\
&= A\vec{y}_t - [\nabla^2 f(A\vec{y}_t)]^{-1}\nabla f(A\vec{y}_t) && (152)\\
&= \vec{x}_t - [\nabla^2 f(\vec{x}_t)]^{-1}\nabla f(\vec{x}_t) && (153)\\
&= \vec{x}_{t+1}. && (154)
\end{aligned}$$

Here the penultimate step replacing $A\vec{y}_t$ by $\vec{x}_t$ is by the inductive hypothesis.

14. Error of LASSO Regression (24 pts)

Let $X \in \mathbb{R}^{n\times d}$ be a matrix, let $\vec{\alpha}_0 \in \mathbb{R}^d$ and $\vec{w} \in \mathbb{R}^n$ be vectors, and let $\lambda > 0$ be a scalar. Suppose that $\vec{y} \in \mathbb{R}^n$ is defined as

$$\vec{y} \doteq X\vec{\alpha}_0 + \vec{w}. \tag{155}$$

Given X and $\vec{y}$, we want to recover $\vec{\alpha}_0$ via LASSO regression. We find an estimate $\vec{\alpha}^\star$ which solves the LASSO problem:

$$\vec{\alpha}^\star = \operatorname*{argmin}_{\vec{\alpha}\in\mathbb{R}^d} f(\vec{\alpha}), \qquad \text{where} \qquad f(\vec{\alpha}) \doteq \frac{1}{2n}\|\vec{y} - X\vec{\alpha}\|_2^2 + \lambda\|\vec{\alpha}\|_1. \tag{156}$$

Define the error of the estimate $\vec{\alpha}^\star$ to be the vector $\vec{\delta} \in \mathbb{R}^d$, i.e.,

$$\vec{\delta} \doteq \vec{\alpha}^\star - \vec{\alpha}_0. \tag{157}$$

In this problem, we will derive and interpret an upper bound on the squared prediction error $\|X\vec{\delta}\|_2^2 = \|X\vec{\alpha}^\star - X\vec{\alpha}_0\|_2^2$.

(a) (5 pts) **Prove that**

$$\|\vec{y} - X\vec{\alpha}^\star\|_2^2 = \|\vec{w}\|_2^2 + \|X\vec{\delta}\|_2^2 - 2(X^\top\vec{w})^\top\vec{\delta}. \tag{158}$$

HINT: Consider adding and subtracting $X\vec{\alpha}_0$ from the term inside the norm on the left-hand side of (158).

Solution: We have

$$\begin{aligned}
\vec{y} - X\vec{\alpha}^\star &= \vec{y} - X\vec{\alpha}_0 + X\vec{\alpha}_0 - X\vec{\alpha}^\star && (159)\\
&= (\vec{y} - X\vec{\alpha}_0) + X(\vec{\alpha}_0 - \vec{\alpha}^\star) && (160)\\
&= \vec{w} + X(-\vec{\delta}) && (161)\\
&= \vec{w} - X\vec{\delta}. && (162)
\end{aligned}$$

Thus

$$\begin{aligned}
\|\vec{y} - X\vec{\alpha}^\star\|_2^2 &= \|\vec{w} - X\vec{\delta}\|_2^2 && (163)\\
&= \|\vec{w}\|_2^2 + \|X\vec{\delta}\|_2^2 - 2\vec{w}^\top X\vec{\delta} && (164)\\
&= \|\vec{w}\|_2^2 + \|X\vec{\delta}\|_2^2 - 2(X^\top\vec{w})^\top\vec{\delta}. && (165)
\end{aligned}$$

(b) (8 pts) **Using** (158)**, prove that**

$$\frac{1}{2n}\|X\vec{\delta}\|_2^2 \le \left(\frac{X^\top\vec{w}}{n}\right)^\top \vec{\delta} + \lambda(\|\vec{\alpha}_0\|_1 - \|\vec{\alpha}^\star\|_1). \tag{166}$$

HINT: Start by arguing that $f(\vec{\alpha}^\star) \le f(\vec{\alpha}_0)$.

Solution: Since $\vec{\alpha}^\star \in \operatorname{argmin}_{\vec{\alpha}\in\mathbb{R}^d} f(\vec{\alpha})$, we have

$$\frac{1}{2n}\|\vec{y} - X\vec{\alpha}^\star\|_2^2 + \lambda\|\vec{\alpha}^\star\|_1 = f(\vec{\alpha}^\star) \le f(\vec{\alpha}_0) = \frac{1}{2n}\|\vec{w}\|_2^2 + \lambda\|\vec{\alpha}_0\|_1. \tag{167}$$

This, in conjunction with (158) to simplify the left-hand side, obtains

$$\begin{aligned}
&\frac{1}{2n}\|\vec{y} - X\vec{\alpha}^\star\|_2^2 + \lambda\|\vec{\alpha}^\star\|_1 \le \frac{1}{2n}\|\vec{w}\|_2^2 + \lambda\|\vec{\alpha}_0\|_1 && (168)\\
\implies\ &\frac{1}{2n}\|\vec{w}\|_2^2 - \frac{\vec{w}^\top X\vec{\delta}}{n} + \frac{1}{2n}\|X\vec{\delta}\|_2^2 + \lambda\|\vec{\alpha}^\star\|_1 \le \frac{1}{2n}\|\vec{w}\|_2^2 + \lambda\|\vec{\alpha}_0\|_1 && (169)\\
\implies\ &\frac{1}{2n}\|X\vec{\delta}\|_2^2 \le \frac{\vec{w}^\top X\vec{\delta}}{n} + \lambda(\|\vec{\alpha}_0\|_1 - \|\vec{\alpha}^\star\|_1), && (170)
\end{aligned}$$

as desired.

(c) (6 pts) From now on, assume that λ is chosen so that[1]

$$2\left\|\frac{X^{\top}\vec{w}}{n}\right\|_{\infty} \leq \lambda. \tag{171}$$

Using (166) and (171), prove that

$$\frac{1}{2n}\|X\vec{\delta}\|_2^2 \leq \lambda\left(\frac{1}{2}\|\vec{\delta}\|_1 + \|\vec{\alpha}_0\|_1 - \|\vec{\alpha}^{\star}\|_1\right). \tag{172}$$

HINT: Use Hölder's inequality.

Solution: By using Holder's inequality, we have

$$\begin{aligned}\frac{\vec{w}^{\top}X\vec{\delta}}{n} &= \frac{(X^{\top}\vec{w})^{\top}\vec{\delta}}{n} &(173)\\ &\leq \left\|\frac{X^{\top}\vec{w}}{n}\right\|_{\infty}\|\vec{\delta}\|_1 &(174)\\ &\leq \frac{\lambda}{2}\|\vec{\delta}\|_1. &(175)\end{aligned}$$

Thus we are given

$$\begin{aligned}\frac{1}{2n}\|X\vec{\delta}\|_2^2 &\leq \left(\frac{X^{\top}\vec{w}}{n}\right)^{\top}\vec{\delta} + \lambda(\|\vec{\alpha}_0\|_1 - \|\vec{\alpha}^{\star}\|_1) &(176)\\ &\leq \frac{\lambda}{2}\|\vec{\delta}\|_1 + \lambda(\|\vec{\alpha}_0\|_1 - \|\vec{\alpha}^{\star}\|_1) &(177)\\ &= \lambda\left(\frac{1}{2}\|\vec{\delta}\|_1 + \|\vec{\alpha}_0\|_1 - \|\vec{\alpha}^{\star}\|_1\right). &(178)\end{aligned}$$

Solution: In the following part, we claim that we can use (172) to show an inequality (194), given by

$$\frac{1}{n}\|X\vec{\delta}\|_2^2 \leq 12\lambda\|\vec{\alpha}_0\|_1. \tag{194}$$

The proof is a bit involved, but we give it here. It involves an intermediate sub-claim.

Claim:

$$\|\vec{\delta}\|_1 \leq 4\|\vec{\alpha}_0\|_1. \tag{179}$$

Proof of claim. Since squared norms are non-negative, the inequality (172) implies

$$\frac{1}{2}\|\vec{\delta}\|_1 + \|\vec{\alpha}_0\|_1 - \|\vec{\alpha}^{\star}\|_1 \geq 0. \tag{180}$$

Thus

$$\begin{aligned}0 &\leq \frac{1}{2}\|\vec{\delta}\|_1 + \|\vec{\alpha}_0\|_1 - \|\vec{\alpha}^{\star}\|_1 &(181)\\ &= \frac{1}{2}(\|\vec{\alpha}^{\star} - \vec{\alpha}_0\|_1) + \|\vec{\alpha}_0\|_1 - \|\vec{\alpha}^{\star}\|_1 &(182)\\ &\leq \frac{1}{2}\|\vec{\alpha}^{\star}\|_1 + \frac{1}{2}\|\vec{\alpha}_0\|_1 + \|\vec{\alpha}_0\|_1 - \|\vec{\alpha}^{\star}\|_1 &(183)\\ &= -\frac{1}{2}\|\vec{\alpha}^{\star}\|_1 + \frac{3}{2}\|\vec{\alpha}_0\|_1 &(184)\end{aligned}$$

[1] Intuitively, this means that λ is chosen to be larger than the maximum correlation of $\vec{w}$ with any one column ("feature") of X.

$$\implies \|\vec{\alpha}^\star\|_1 \leq 3\|\vec{\alpha}_0\|_1. \tag{185}$$

Now we have

$$\|\vec{\delta}\|_1 = \|\vec{\alpha}^\star - \vec{\alpha}_0\|_1 \leq \|\vec{\alpha}^\star\|_1 + \|\vec{\alpha}_0\|_1 \leq 4\|\vec{\alpha}_0\|_1, \tag{186}$$

as desired.

Back to main proof. We have from (172) that

$$\begin{aligned}
\frac{1}{2n}\|X\vec{\delta}\|_2^2 &\leq \lambda\left(\frac{1}{2}\|\vec{\delta}\|_1 + \{\|\vec{\alpha}_0\|_1 - \|\vec{\alpha}^\star\|_1\}\right) && (187)\\
&\leq \lambda\left(\frac{1}{2}\|\vec{\delta}\|_1 + \|\vec{\alpha}_0 - \vec{\alpha}^\star\|_1\right), \qquad \text{(reverse triangle inequality)} && (188)\\
&= \lambda\left(\frac{1}{2}\|\vec{\delta}\|_1 + \|\vec{\delta}\|_1\right) && (189)\\
&= \frac{3}{2}\lambda\|\vec{\delta}\|_1. && (190)
\end{aligned}$$

Thus, using (179), we have

$$\frac{1}{n}\|X\vec{\delta}\|_2^2 \leq 3\lambda\|\vec{\delta}\|_1 \leq 12\lambda\|\vec{\alpha}_0\|_1 \tag{191}$$

as desired.

Reverse triangle inequality. The very last loose end we have is the so-called reverse triangle inequality, which states

$$\|\vec{u}\|_1 - \|\vec{v}\|_1 \leq \|\vec{u} - \vec{v}\|_1, \qquad \text{for all } \vec{u}, \vec{v} \in \mathbb{R}^d. \tag{192}$$

Indeed this holds because we write

$$\|\vec{u}\|_1 = \|\vec{v} + (\vec{u} - \vec{v})\|_1 \leq \|\vec{v}\|_1 + \|\vec{u} - \vec{v}\|_1 \tag{193}$$

and rearrange.

(d) (5 pts) The results of parts 14(a), 14(b), and 14(c) can be used to show that if $\lambda \geq 2\|X^\top \vec{w}/n\|_\infty$ then

$$\frac{1}{n}\|X\vec{\delta}\|_2^2 \leq 12\lambda\|\vec{\alpha}_0\|_1. \tag{194}$$

In this subpart, assume (194) holds — *you do not need to prove it*. Suppose that $\vec{\alpha}_0$ has $s > 0$ nonzero entries, i.e., $\|\vec{\alpha}_0\|_0 = s$. (Here $\|\cdot\|_0$ is the ℓ^0 "norm" which counts the number of nonzero entries of its input).

i. **Using (194), prove that**

$$\frac{1}{n}\|X\vec{\delta}\|_2^2 \leq 12\lambda\sqrt{s}\|\vec{\alpha}_0\|_2. \tag{195}$$

HINT: Recall the inequality (proved in homework, and thus usable without proof) that

$$\|\vec{x}\|_1 \leq \sqrt{\|\vec{x}\|_0} \cdot \|\vec{x}\|_2, \qquad \textit{for all } \vec{x} \in \mathbb{R}^d. \tag{196}$$

ii. Suppose that $\|\vec{\alpha}_0\|_2 = 1$ and $\lambda = 1$ are fixed. **How does the right-hand side of (195) grow in the high-dimensional limit $d \to \infty$ if $s = \sqrt{d}$?**

iii. Suppose that $\|\vec{\alpha}_0\|_2 = 1$ and $\lambda = 1$ are fixed. **How does the right-hand side of (195) grow in the high-dimensional limit $d \to \infty$ if $s = 1$?**

The result of this problem shows that the prediction error $\|X\vec{\delta}\|_2^2$ is bounded, even when arbitrarily many spurious features are added (i.e., $d \to \infty$), when using a LASSO estimator.

Solution: We apply the bound (196) in one line:

$$\frac{\|X\vec{\delta}\|_2^2}{n} \leq 12\lambda\|\vec{\alpha}_0\|_1 \leq 12\lambda\sqrt{\|\vec{\alpha}_0\|_0} \cdot \|\vec{\alpha}_0\|_2 = 12\lambda\sqrt{s}\|\vec{\alpha}_0\|_2. \tag{197}$$

If $s = \sqrt{d}$, then the right-hand side of this inequality grows as $d^{1/4}$, hence growing to ∞ as $d \to \infty$. If $s = 1$, then the right-hand side of this inequality is constant, and remains a constant as $d \to \infty$.

Some general discussion. If we don't have any assumptions on the sparsity of $\vec{\alpha}_0$, then in general, we would only be able to do as good as the rate

$$\frac{\|X\vec{\delta}\|_2^2}{n} \leq 12\lambda\sqrt{d}\|\vec{\alpha}_0\|_2, \tag{198}$$

meaning that our error scales with $\sqrt{d}$. However, if $\vec{\alpha}_0$ is a very high-dimensional vector (i.e., $d \gg n$) with few, say s, nonzero entries, then the real error rate only depends on $\sqrt{s}$. This is much better as it reduces the error bound dramatically. (Statisticians would call this bound "non-vacuous in the high-dimensional regime", whereas the other bound is "vacuous in the high-dimensional regime" because the right-hand side goes to ∞ as $d \to \infty$).

The result makes sense in the context of the LASSO objective because the $\|\cdot\|_1$ regularization encourages sparsity in the solution vector, and if the true solution is indeed sparse, it is able to be recovered much more effectively by LASSO solvers (at least up to multiplication by X).

