

1. Eigenvalues of Symmetric Matrices

- (a) Let $A \in \mathbb{S}^n$ (i.e., the set of $n \times n$ real symmetric matrices) with eigenvalues λ_i . Prove that all of the eigenvalues of A are real (i.e. that $\lambda_i \in \mathbb{R}$ for each i).

Solution: Let $(v \in \mathbb{C}^n, \lambda \in \mathbb{C})$ be an eigen-pair of the matrix A . Consider the quantity $(Av)^*v$. We have that

$$(Av)^*v = v^*A^*v = v^*Av = \lambda v^*v$$

where the second equality uses the fact that A is symmetric and real and the third that v is an eigenvector of A with eigenvalue λ .

Similarly, we can apply the fact that $Av = \lambda v$ before expanding the inner product to get

$$(Av)^*v = (\lambda v)^*v = \bar{\lambda}v^*v$$

where $\bar{\lambda}$ notates the complex conjugate of λ . As a result,

$$\lambda v^*v = \bar{\lambda}v^*v$$

Since v is a non-zero vector, $v^*v = \|v\|_2^2 > 0$, so we can re-arrange our equality to see that

$$\lambda = \bar{\lambda}$$

The only numbers which are equal to their own complex conjugate are the real numbers, so this tells us that $\lambda \in \mathbb{R}$ as desired.

- (b) Let $A \in \mathbb{S}^n$ (i.e. the set of $n \times n$ real symmetric matrices) and $(\lambda_1, \vec{u}_1), (\lambda_2, \vec{u}_2), \lambda_1 \neq \lambda_2$ be distinct eigen-pairs of A . Show that $\vec{u}_1^\top \vec{u}_2 = 0$, i.e., eigenspaces corresponding to distinct eigenvalues are mutually orthogonal.

Solution: We have

$$\lambda_1 \vec{u}_1^\top \vec{u}_2 = (\lambda_1 \vec{u}_1)^\top \vec{u}_2 \tag{1}$$

$$= (A\vec{u}_1)^\top \vec{u}_2 \tag{2}$$

$$= \vec{u}_1^\top A^\top \vec{u}_2 \tag{3}$$

$$= \vec{u}_1^\top A \vec{u}_2 \tag{4}$$

$$= \vec{u}_1^\top (\lambda_2 \vec{u}_2) \tag{5}$$

$$= \lambda_2 \vec{u}_1^\top \vec{u}_2. \tag{6}$$

Thus we have

$$\lambda_1(\vec{u}_1^\top \vec{u}_2) = \lambda_2(\vec{u}_1^\top \vec{u}_2) \implies (\lambda_1 - \lambda_2)(\vec{u}_1^\top \vec{u}_2) = 0. \tag{7}$$

Since $\lambda_1 \neq \lambda_2$, we have $\lambda_1 - \lambda_2 \neq 0$, so we must have $\vec{u}_1^\top \vec{u}_2 = 0$.

Thus, $\vec{u}_1^\top \vec{u}_2 = 0$ for any \vec{u}_1, \vec{u}_2 corresponding to different eigenvalues. Stated differently, unique eigenvalues correspond to orthogonal eigenvectors.