Quiz01

1. Eigenvalues of Symmetric Matrices

(a) Let $A \in \mathbb{S}^n$ (i.e., the set of $n \times n$ real symmetric matrices) with eigenvalues λ_i. Prove that all of the eigenvalues of A are real (i.e. that $\lambda_i \in \mathbb{R}$ for each i).

Solution: Let $(v \in \mathbb{C}^n, \lambda \in \mathbb{C})$ be an eigen-pair of the matrix A. Consider the quantity $(Av)^* v$. We have that

$$(Av)^* v = v^* A^* v = v^* A v = \lambda v^* v$$

where the second equality uses the fact that A is symmetric and real and the third that v is an eigenvector of A with eigenvalue λ.

Similarly, we can apply the fact that $Av = \lambda v$ before expanding the inner product to get

$$(Av)^* v = (\lambda v)^* v = \overline{\lambda} v^* v$$

where $\overline{\lambda}$ notates the complex conjugate of λ. As a result,

$$\lambda v^* v = \overline{\lambda} v^* v$$

Since v is a non-zero vector, $v^* v = \|v\|_2^2 > 0$, so we can re-arrange our equality to see that

$$\lambda = \overline{\lambda}$$

The only numbers which are equal to their own complex conjugate are the real numbers, so this tells us that $\lambda \in \mathbb{R}$ as desired.

(b) Let $A \in \mathbb{S}^n$ (i.e. the set of $n \times n$ real symmetric matrices) and $(\lambda_1, \vec{u}_1), (\lambda_2, \vec{u}_2), \lambda_1 \neq \lambda_2$ be distinct eigen-pairs of A. Show that $\vec{u}_1^\top \vec{u}_2 = 0$, i.e., eigenspaces corresponding to distinct eigenvalues are mutually orthogonal.

Solution: We have

$$\begin{aligned}
\lambda_1 \vec{u}_1^\top \vec{u}_2 &= (\lambda_1 \vec{u}_1)^\top \vec{u}_2 && (1)\\
&= (A\vec{u}_1)^\top \vec{u}_2 && (2)\\
&= \vec{u}_1^\top A^\top \vec{u}_2 && (3)\\
&= \vec{u}_1^\top A \vec{u}_2 && (4)\\
&= \vec{u}_1^\top (\lambda_2 \vec{u}_2) && (5)\\
&= \lambda_2 \vec{u}_1^\top \vec{u}_2. && (6)
\end{aligned}$$

Thus we have

$$\lambda_1(\vec{u}_1^\top \vec{u}_2) = \lambda_2(\vec{u}_1^\top \vec{u}_2) \implies (\lambda_1 - \lambda_2)(\vec{u}_1^\top \vec{u}_2) = 0. \tag{7}$$

Since $\lambda_1 \neq \lambda_2$, we have $\lambda_1 - \lambda_2 \neq 0$, so we must have $\vec{u}_1^\top \vec{u}_2 = 0$.

Thus, $\vec{u}_1^\top \vec{u}_2 = 0$ for any $\vec{u}_1, \vec{u}_2$ corresponding to different eigenvalues. Stated differently, unique eigenvalues correspond to orthogonal eigenvectors.