

1. SVD Part 2

Consider A to be the 4×3 matrix

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \quad (1)$$

where \vec{a}_i for $i \in \{1, 2, 3\}$ form a set of *orthogonal* vectors satisfying $\|\vec{a}_1\|_2 = 3$, $\|\vec{a}_2\|_2 = 2$, $\|\vec{a}_3\|_2 = 1$.

- (a) What is the compact SVD of A ? Express it as $A = U\Sigma V^\top$, with Σ the diagonal matrix of singular values ordered in decreasing fashion, and explicitly describe U and V .

Solution: The SVD of $A = U\Sigma V^\top$. Due to the orthogonality of the \vec{a}_i we have that

$$A^\top A = V\Sigma^2 V^\top = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2)$$

Thus $V = I$ and $\Sigma = \text{diag}(3, 2, 1)$. Finally we have that $U = A\Sigma^{-1}$ which becomes

$$U = \begin{bmatrix} \frac{\vec{a}_1}{3} & \frac{\vec{a}_2}{2} & \frac{\vec{a}_3}{1} \end{bmatrix} \quad (3)$$

- (b) What is the dimension of the null space, $\dim(\mathcal{N}(A))$?

Solution: From part (a) all of the singular values of the A are non-zero. So the dimension of the null space is 0. Alternatively, all the columns of A are orthogonal – so no (non-zero) linear combination of them can equal zero.

- (c) What is the rank of A , $\text{rank}(A)$? Provide an orthonormal basis for the range of A .

Solution: The rank of A is simply the number of non-zero singular values. So $\text{rank}(A) = 3$. The columns of U (defined above) provide an orthonormal basis for the range of A .

- (d) Let I_3 denote the 3×3 identity matrix. Consider the matrix $\tilde{A} = \begin{bmatrix} A \\ I_3 \end{bmatrix} \in \mathbb{R}^{7 \times 3}$. What are the singular values of \tilde{A} (in terms of the singular values of A)?

Solution: We have that $\tilde{A}^\top \tilde{A} = A^\top A + I_3 = V(\Sigma^2 + I_3)V^\top$. Hence if we denote σ_i as the singular values of A then the singular values of \tilde{A} are $\tilde{\sigma}_i = \sqrt{\sigma_i^2 + 1}$ which are $\sqrt{10}, \sqrt{5}, \sqrt{2}$.