



### 1. Gradients, Jacobians, and Hessians

Suppose  $A \in \mathbb{R}^{n \times n}$  is a square matrix whose entries are denoted  $a_{ij}$  and whose rows are denoted  $\vec{a}_1^\top, \dots, \vec{a}_n^\top$ , and  $\vec{b} \in \mathbb{R}^n$  is a vector whose entries are denoted  $b_i$ .

Compute the gradients for the following functions. Show your work.

(a)  $g_1(\vec{x}) = \vec{x}^\top A \vec{x}$ .

**Solution:** We have

$$g_1(\vec{x}) = \vec{x}^\top A \vec{x} \quad (1)$$

$$= \vec{x}^\top \begin{bmatrix} \vec{a}_1^\top \\ \vdots \\ \vec{a}_n^\top \end{bmatrix} \vec{x} \quad (2)$$

$$= \vec{x}^\top \begin{bmatrix} \vec{a}_1^\top \vec{x} \\ \vdots \\ \vec{a}_n^\top \vec{x} \end{bmatrix} \quad (3)$$

$$= \sum_{i=1}^n x_i (\vec{a}_i^\top \vec{x}). \quad (4)$$

Taking the derivative with respect to any  $x_j$  yields

$$\frac{\partial g_1}{\partial x_j}(\vec{x}) = \frac{\partial}{\partial x_j} \sum_{i=1}^n x_i (\vec{a}_i^\top \vec{x}) \quad (5)$$

$$= \frac{\partial}{\partial x_j} \left[ x_j (\vec{a}_j^\top \vec{x}) + \sum_{\substack{i=1 \\ i \neq j}}^n x_i (\vec{a}_i^\top \vec{x}) \right] \quad (6)$$

$$= \frac{\partial}{\partial x_j} x_j (\vec{a}_j^\top \vec{x}) + \sum_{\substack{i=1 \\ i \neq j}}^n \frac{\partial}{\partial x_j} x_i (\vec{a}_i^\top \vec{x}) \quad (7)$$

$$= x_j \frac{\partial}{\partial x_j} (\vec{a}_j^\top \vec{x}) + \vec{a}_j^\top \vec{x} + \sum_{\substack{i=1 \\ i \neq j}}^n x_i \frac{\partial}{\partial x_j} (\vec{a}_i^\top \vec{x}) \quad (8)$$

$$= x_j (\vec{a}_j)_j + \vec{a}_j^\top \vec{x} + \sum_{\substack{i=1 \\ i \neq j}}^n x_i (\vec{a}_i)_j \quad (9)$$

$$= \vec{a}_j^\top \vec{x} + \sum_{i=1}^n x_i (\vec{a}_i)_j \quad (10)$$

$$= \sum_{i=1}^n a_{ji} x_i + \sum_{i=1}^n a_{ij} x_i \quad (11)$$

$$= (A \vec{x})_j + (A^\top \vec{x})_j \quad (12)$$

$$= [(A + A^\top) \vec{x}]_j. \quad (13)$$

Thus we have

$$\nabla g_1(\vec{x}) = (A + A^\top) \vec{x}. \quad (14)$$

(b)  $g_6(\vec{x}) = e^{\|\vec{x}\|_2^2}$ .

**Solution:** We use the multivariable chain rule, obtaining

$$\nabla g_6(\vec{x}) = (Dg_6(\vec{x}))^\top \quad (15)$$

$$= ([D \exp(g_2(\vec{x}))][Dg_2(\vec{x})])^\top \quad (16)$$

$$= [Dg_2(\vec{x})]^\top [D \exp(g_2(\vec{x}))]^\top \quad (17)$$

$$= [\nabla g_2(\vec{x})][e^{g_2(\vec{x})}]^\top \quad (18)$$

$$= [2\vec{x}][e^{g_2(\vec{x})}] \quad (19)$$

$$= 2e^{\|\vec{x}\|_2^2}\vec{x}, \quad (20)$$

since the transpose of a scalar is a scalar.

(c)  $g_7(\vec{x}) = e^{\|A\vec{x} - \vec{b}\|_2^2} = g_6(A\vec{x} - \vec{b})$ .

**Solution:** Notice that in part ??, we did not use any specific functional properties of  $g_4$ ,  $\nabla g_4$ , or  $\nabla^2 g_5$  to determine  $\nabla g_5$  and  $\nabla^2 g_5$ . Using the exact same analysis, we can conclude that

$$\nabla g_7 = A^\top [\nabla g_6(A\vec{x} - \vec{b})] \quad (21)$$