Quiz02

1. Singular Values

(a) Suppose $A \in \mathbb{R}^{3\times 2}$ is a matrix such that $A^\top A$ is given by

$$A^\top A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}. \tag{1}$$

What are the singular values of A?

Solution: The singular values are the square roots of the eigenvalues of $A^\top A$. The eigenvalues of $A^\top A$ are 5 and 3, since those are the diagonal entries of the diagonal matrix in the spectral decomposition.

Therefore, the singular values of A are $\sqrt{5}$ and $\sqrt{3}$.

(b) Suppose that $B \in \mathbb{R}^{3\times 2}$ has singular values $\sqrt{2}$ and $\sqrt{7}$. Let $C = \begin{bmatrix} B & -B & 3I_3 \end{bmatrix} \in \mathbb{R}^{3\times 7}$, where $I_3 \in \mathbb{R}^{3\times 3}$ is the 3×3 identity matrix. **What are the singular values of C?**

HINT: Consider the matrix $CC^\top \in \mathbb{R}^{3\times 3}$.

Solution: First, observe that for an arbitrary matrix A, the nonzero eigenvalues of $A^\top A$ (corresponding to the squares of the nonzero singular values of A) are equal to the nonzero eigenvalues of $AA^\top$. This is easy to show using the SVD of A. We will use this result twice, once for B and once for C.

To find the singular values of C, we consider $CC^\top \in \mathbb{R}^{3\times 3}$. Note that we consider this matrix rather than $C^\top C \in \mathbb{R}^{7\times 7}$ because the former is smaller, and we get the following simplification:

$$CC^\top = \begin{bmatrix} B & -B & 3I \end{bmatrix} \begin{bmatrix} B^\top \\ -B^\top \\ 3I \end{bmatrix} = 2BB^\top + 9I. \tag{2}$$

By the shift and scale properties of eigenvalues, the eigenvalues of $CC^\top$ are $9 + 2\times$ the eigenvalues of $BB^\top$. Since the nonzero eigenvalues of $BB^\top$ are the squared nonzero singular values of B (with the same geometric multiplicity), we know that the nonzero eigenvalues of $BB^\top$ are 2 and 7. In addition, since $BB^\top$ is a 3×3 matrix and is diagonalizable by the spectral theorem, i.e., the geometric multiplicities of its eigenvalues sum to 3, it must also have 0 as a third eigenvalue, with multiplicity 1. Thus the eigenvalues of $CC^\top$ are 9, 13 and 23. Thus the nonzero singular values of C are 3, $\sqrt{13}$, and $\sqrt{23}$.