Quiz02

1. Gram Schmidt

(a) Given a matrix $A \in \mathbb{R}^{n\times n}$, it could be represented as a multiplication of two matrices

$$A = QR, \tag{1}$$

where $Q \in \mathbb{R}^{n\times n}$ is an orthonormal matrix and $R \in \mathbb{R}^{n\times n}$ is an upper-triangular matrix. For the matrix A, describe how Gram-Schmidt process could be used to find the Q and R matrices, and apply this to

$$A = \begin{bmatrix} 3 & -3 & 1 \\ 4 & -4 & -7 \\ 0 & 3 & 3 \end{bmatrix} \tag{2}$$

to find an orthonormal matrix Q and an upper-triangular matrix R.

Solution: Let $\vec{a}_i$ and $\vec{q}_i$ denote the columns of A and Q, respectively. Using Gram-Schmidt, we obtain an orthonormal basis $\vec{q}_i$ for the column space of A.

$$\vec{p}_1 = \vec{a}_1, \vec{q}_1 = \frac{\vec{p}_1}{\|\vec{p}_1\|_2} \tag{3}$$

$$\vec{p}_2 = \vec{a}_2 - (\vec{a}_2^\top \vec{q}_1)\vec{q}_1, \quad \vec{q}_2 = \frac{\vec{p}_2}{\|\vec{p}_2\|_2} \tag{4}$$

$$\vec{p}_3 = \vec{a}_3 - (\vec{a}_3^\top \vec{q}_1)\vec{q}_1 - (\vec{a}_3^\top \vec{q}_2)\vec{q}_2, \vec{q}_3 = \frac{\vec{p}_3}{\|\vec{p}_3\|_2} \tag{5}$$

$$\vdots \tag{6}$$

Rearranging terms, we have

$$\vec{a}_1 = r_{11}\vec{q}_1 \tag{7a}$$

$$\vec{a}_i = r_{i1}\vec{q}_1 + \cdots + r_{ii}\vec{q}_i, \quad i = 2, ..., n, \tag{7b}$$

where each $\vec{q}_i$ has unit norm, and $r_{ij}\vec{q}_j$ denotes the projection of $\vec{a}_i$ onto the vector $\vec{q}_j$ for $j \neq i$.

Stacking $\vec{a}_i$ horizontally into A and rewriting (7a-b) in matrix notation, we obtain $A = QR$. For the given matrix, we have

$$A = \begin{bmatrix} 0.6 & 0 & 0.8 \\ 0.8 & 0 & -0.6 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & -5 & -5 \\ 0 & 3 & 3 \\ 0 & 0 & 5 \end{bmatrix}. \tag{8}$$

Note that an equivalent factorization is $A = (-Q)(-R)$.

(b) Given an invertible matrix $A \in \mathbb{R}^{n\times n}$ and an observation vector $\vec{b} \in \mathbb{R}^n$, the solution to the equality

$$A\vec{x} = \vec{b} \tag{9}$$

is given as $\vec{x} = A^{-1}\vec{b}$. For the matrix $A = QR$ from part 1(a), assume that we want to solve

$$A\vec{x} = \begin{bmatrix} 8 \\ -6 \\ 3 \end{bmatrix}. \tag{10}$$

By using the fact that Q is an orthonormal matrix, find $\vec{v}$ such that

$$R\vec{x} = \vec{v}. \tag{11}$$

Solution: We note that $Q^{-1} = Q^\top$.

$$A\vec{x} = \vec{b} \tag{12}$$

$$QR\vec{x} = \vec{b} \tag{13}$$

$$Q^\top QR\vec{x} = R\vec{x} = Q^\top\vec{b}. \tag{14}$$

Thus

$$\vec{v} = Q^\top\vec{b} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix}. \tag{15}$$