

1. Restriction to a line

Show that a function f is convex if and only if for all $\vec{x} \in \text{dom}(f)$ and all \vec{v} , the function $g : \text{dom}(g) \rightarrow \mathbb{R}$ given by $g(t) = f(\vec{x} + t\vec{v})$ is convex for $\text{dom}(g) = \{t \in \mathbb{R} \mid \vec{x} + t\vec{v} \in \text{dom}(f)\}$.

Solution: In the first direction: assume f is convex and consider $\vec{x} \in \text{dom}(f)$, \vec{v} and the function $g : \text{dom}(g) \rightarrow \mathbb{R}$ given by $g(t) = f(\vec{x} + t\vec{v})$ where $\text{dom}(g) = \{t \in \mathbb{R} \mid \vec{x} + t\vec{v} \in \text{dom}(f)\}$.

Because f is convex, $\text{dom}(f)$ is convex, therefore $\text{dom}(g)$ is also convex. For $t_1, t_2 \in \text{dom}(g)$ and $\lambda \in [0, 1]$:

$$g(\lambda t_1 + (1 - \lambda)t_2) = f(\vec{x} + (\lambda t_1 + (1 - \lambda)t_2)\vec{v}) \quad (1)$$

$$= f(\lambda(\vec{x} + t_1\vec{v}) + (1 - \lambda)(\vec{x} + t_2\vec{v})) \quad (2)$$

$$\leq \lambda f(\vec{x} + t_1\vec{v}) + (1 - \lambda)f(\vec{x} + t_2\vec{v}) \quad (3)$$

$$= \lambda g(t_1) + (1 - \lambda)g(t_2) \quad (4)$$

Therefore g is convex.

In the other direction: Consider $\vec{x}_1, \vec{x}_2 \in \text{dom}(f)$ and $\lambda \in [0, 1]$. Define $g : t \rightarrow f(\vec{x}_2 + t(\vec{x}_1 - \vec{x}_2))$. g is convex and $0 \in \text{dom}(g)$ and $1 \in \text{dom}(g)$, so $[0, 1] \in \text{dom}(g)$. Therefore $\lambda\vec{x}_1 + (1 - \lambda)\vec{x}_2 \in \text{dom}(f)$ and $\text{dom}(f)$ is convex.

Because g is convex:

$$g(\lambda 1 + (1 - \lambda)0) = g(\lambda) \leq \lambda g(1) + (1 - \lambda)g(0) \quad (5)$$

$$f(\vec{x}_2 + \lambda(\vec{x}_1 - \vec{x}_2)) \leq \lambda f(\vec{x}_2 + 1(\vec{x}_1 - \vec{x}_2)) + (1 - \lambda)f(\vec{x}_2 + 0(\vec{x}_1 - \vec{x}_2)) \quad (6)$$

$$f(\lambda\vec{x}_1 + (1 - \lambda)\vec{x}_2) \leq \lambda f(\vec{x}_1) + (1 - \lambda)f(\vec{x}_2) \quad (7)$$

Therefore f is convex.