Quiz03

1. PSD Matrices

In this problem, we will analyze properties of positive semidefinite (PSD) matrices. A *symmetric* matrix $M \in \mathbb{R}^{n\times n}$ is a PSD matrix if $\vec{x}^\top M\vec{x} \geq 0$ for all $\vec{x} \in \mathbb{R}^n$, and we denote that as $M \succeq 0$ or $M \in \mathbb{S}^n_+$.

Assume $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix.

(a) Show that $A \succeq 0$ if and only if all eigenvalues of A are non-negative.

Solution: $\implies$:

i. Solution 1: We can plug in the Spectral Decomposition here:

$$\vec{x}^\top A\vec{x} = \vec{x}^\top U\Sigma U^\top \vec{x} = \vec{v}^\top \Sigma \vec{v} \geq 0, \tag{1}$$

where $\vec{v} := U^\top \vec{x}$ is a rotated version of $\vec{x}$ since U is orthonormal. Now, we just need to convert that final quadratic into any eigenvalue of A, and we can do that by choosing a $\vec{v}$ that pulls out whichever eigenvalue we want (e.g. if we want the first eigenvalue, we can choose the first unit vector). To be thorough, we can then realize that the set of $\vec{x}$'s such that $U^\top \vec{x} = \vec{e}_i$ for any unit vector, will pull out the ith eigenvalue, thus satisfying definition 2.

ii. Solution 2: We can just use the definition of an eigenvalue:

$$\vec{x}^\top A\vec{x} = \vec{x}\lambda\vec{x} = \lambda\vec{x}^\top\vec{x} = \lambda \left\|\vec{x}\right\|_2^2 \tag{2}$$

Since norms/anything squared is always non-negative, in order for $\lambda \left\|\vec{x}\right\|_2^2 \geq 0$, λ must be non-negative.

$\impliedby$: Using the Spectral Decomposition again, we arrive at the equation $\vec{v}^\top\Sigma\vec{v}$, which we can expand further:

$$\vec{v}^\top\Sigma\vec{v} = \sum_i \lambda_i v_i^2 \geq 0, \tag{3}$$

where the last inequality came from the fact that anything squared is non-negative and all eigenvalues are non-negative by assumption of the problem.

(b) Show that if $A \succeq 0$ then all diagonal entries of A are non-negative, $A_{ii} \geq 0$.

Solution: The quadratic form $\vec{x}^\top A\vec{x} \geq 0$ applies for all vectors $\vec{x}$. Therefore, let's choose a vector that will pull out A_{ii}: the ith unit vector. $A\vec{e}_i$ pulls out the ith column $\vec{a}_i$, followed by $\vec{e}_i^\top \vec{a}_i$, which will pull out the ith element of the ith column. Therefore, $\vec{e}_i^\top A\vec{e}_i = A_{ii} \geq 0$.

(c) Show that if $A \in \mathbb{S}^n_+$, then there exists $P \in \mathbb{S}^n_+$ such that $A = P^2$.

Solution: Since A is symmetric positive semidefinite, A has non-negative eigenvalues. Thus, we can diagonalize A as $A = U\Sigma U^\top$, where the diagonal matrix of eigenvalues Σ has all non-negative entries on the diagonal. Then, we are able to define a matrix $A^{\frac{1}{2}} = U\Sigma^{\frac{1}{2}}U^\top$, where $\Sigma^{\frac{1}{2}}$ is a diagonal matrix with the square roots of A's eigenvalues. Note that $A^{\frac{1}{2}}$ is PSD since its eigenvalues are still non-negative. Thus, with $P = A^{\frac{1}{2}}$, we can show the following:

$$P^\top P = (A^{\frac{1}{2}})^\top A^{\frac{1}{2}} = (U\Sigma^{\frac{1}{2}}U^\top)^\top U\Sigma^{\frac{1}{2}}U^\top = U\Sigma^{\frac{1}{2}}U^\top U\Sigma^{\frac{1}{2}}U^\top = U\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}U^\top = U\Sigma U^\top = A. \tag{4}$$