Quiz03

1. Perspective function

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function, with domain the entire space $\mathbb{R}^n$.

(a) Show that we can represent f as a pointwise maximum of affine functions, specifically

$$\forall\, x \in \mathbb{R}^n \;:\; f(x) = \max_{(a,b)\in\mathcal{A}} \; a^T x + b, \tag{1}$$

where $\mathcal{A} \subseteq \mathbb{R}^{n+1}$ is to be determined.

Hint: For every $x, y \in \mathbb{R}^n$: $f(x) \geq f(y) + (x-y)^T \nabla f(y)$.

Solution: Let $x \in \mathbb{R}^n$. We have

$$\forall\, y \in \mathbb{R}^n \;:\; f(x) \geq f(y) + (x-y)^T \nabla f(y).$$

Maximizing over y:

$$f(x) \geq \max_y \; f(y) + (x-y)^T \nabla f(y).$$

Choosing $y = x$ in the expression inside the maximum results in $f(x)$. Hence, the maximum over y is larger than $f(x)$, which leads to the equality:

$$f(x) = \max_y \; f(y) + (x-y)^T \nabla f(y).$$

We thus choose

$$\mathcal{A} = \left\{ (a,b) \in \mathbb{R}^n \times \mathbb{R} \;:\; a = \nabla f(y), \;\; b = f(y) - y^T \nabla f(y), \;\; y \in \mathbb{R}^n \right\}.$$

(b) We define the *perspective* of f as the function $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ with values

$$g(x,t) = \begin{cases} t f(x/t) & \text{if } t > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Prove that g is convex.

Hint: Use (1).

Solution: The domain of g is $\mathbb{R}^n \times \mathbb{R}_{++}$, so it is convex. When $t > 0$:

$$g(x,t) = t \max_{(a,b)\in\mathcal{A}} \; a^T(x/t) + b = \max_{(a,b)\in\mathcal{A}} \; a^T x + tb.$$

We observe that g is the point-wise maximum of linear functions of (x,t), hence it is convex.

(c) Show that the function $h : \mathbb{R}^n \to \mathbb{R}$ with values

$$h(x) = \begin{cases} \dfrac{(a^T x + b)^2}{c^T x + d} & \text{if } c^T x + d > 0, \\ +\infty & \text{otherwise,} \end{cases}$$

is convex.

Hint: Consider the perspective function of $f(z) = z^2$ on $\mathbb{R}$.

Solution: The function is the composition of the affine map

$$x \to (z,t) = (a^T x + b, c^T x + d)$$

with the function $(z,t) \to z^2/t$ on $\mathbb{R} \times \mathbb{R}_{++}$. The latter turns out to be the perspective of the function f with values $f(z) = z^2$ on $\mathbb{R}$. Since f can be represented as a pointwise maximum of affine function, we obtain that its perspective is convex.