



### 1. Lagrangian Dual of a QP

Consider the standard form of a convex quadratic program, with  $Q \succ 0$ :

$$\min_{\vec{x}} \quad \frac{1}{2} \vec{x}^\top Q \vec{x} \quad (1)$$

$$\text{s.t.} \quad A \vec{x} \leq \vec{b} \quad (2)$$

(a) Write the Lagrangian function  $\mathcal{L}(\vec{x}, \vec{\lambda})$ .

**Solution:**

$$\mathcal{L}(\vec{x}, \vec{\lambda}) = \frac{1}{2} \vec{x}^\top Q \vec{x} + \vec{\lambda}^\top (A \vec{x} - \vec{b}) \quad (3)$$

(b) Write the Lagrangian dual function,  $g(\vec{\lambda})$ .

**Solution:**

$$g(\vec{\lambda}) = \inf_{\vec{x}} \mathcal{L}(\vec{x}, \vec{\lambda}) \quad (4)$$

We can find this infimum by setting  $\nabla_{\vec{x}} \mathcal{L}(\vec{x}^*, \vec{\lambda}) = 0$ :

$$Q \vec{x}^* + A^\top \vec{\lambda} = 0 \implies \vec{x}^* = -Q^{-1} A^\top \vec{\lambda} \quad (5)$$

Substituting, we get

$$g(\vec{\lambda}) = \mathcal{L}(\vec{x}^*, \vec{\lambda}) \quad (6)$$

$$= \frac{1}{2} \vec{\lambda}^\top A Q^{-1} A^\top \vec{\lambda} - \vec{\lambda}^\top A Q^{-1} A^\top \vec{\lambda} - \vec{\lambda}^\top \vec{b} \quad (7)$$

$$= -\frac{1}{2} \vec{\lambda}^\top A Q^{-1} A^\top \vec{\lambda} - \vec{\lambda}^\top \vec{b} \quad (8)$$

(c) Show that the Lagrangian dual problem is convex by writing it in “standard QP form” – that is, in a form similar to the original problem. Is the Lagrangian dual problem convex in general?

**Solution:** The Lagrangian dual problem writes

$$\max_{\vec{\lambda} \geq 0} g(\vec{\lambda}) = \max_{\vec{\lambda} \geq 0} -\frac{1}{2} \vec{\lambda}^\top A Q^{-1} A^\top \vec{\lambda} - \vec{\lambda}^\top \vec{b}, \quad (9)$$

the maximization of a concave function of  $\vec{\lambda}$  over the convex region given by the non-negative orthant  $\vec{\lambda} \geq 0$ . The dual problem is therefore convex.

While in this problem, the primal problem was convex, it turns out that the Lagrangian dual problem is a convex problem even when the primal is not. To see this, examine its general form:

$$\max_{\vec{\lambda} \geq 0} \min_{\vec{x}} \mathcal{L}(\vec{x}, \vec{\lambda}) = \max_{\vec{\lambda} \geq 0} \min_{\vec{x}} \left[ f_0(\vec{x}) + \sum_{i=1}^n \lambda_i f_i(\vec{x}) \right] \quad (10)$$

This represents the pointwise minimum of affine functions of  $\vec{\lambda}$ , which we know to be concave. The resulting maximization problem of a concave objective in  $\vec{\lambda}$  over the convex region  $\vec{\lambda} \geq 0$  is then a convex optimization problem!