Quiz03

1. Optimizing Over Multiple Variables

In this exercise, we consider several problems in which we optimize over two variables, $\vec{x} \in \mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^m$, and a general (possibly nonconvex) objective function, $F_0(\vec{x}, \vec{y})$. Suppose also that $\vec{x}$ and $\vec{y}$ are constrained to different feasible sets $\mathcal{X}$ and $\mathcal{Y}$, respectively, which may or may not be convex.

(a) Show that

$$\min_{\vec{x} \in \mathcal{X}} \min_{\vec{y} \in \mathcal{Y}} F_0(\vec{x}, \vec{y}) = \min_{\vec{y} \in \mathcal{Y}} \min_{\vec{x} \in \mathcal{X}} F_0(\vec{x}, \vec{y}), \tag{1}$$

i.e., if we minimize over both $\vec{x}$ and $\vec{y}$, then we can exchange the minimization order without altering the optimal value.

Solution: We first consider the quantity $\min_{\vec{y} \in \mathcal{Y}} F_0(\vec{x}, \vec{y})$, which can be viewed as a function of $\vec{x}$. We can write

$$F_0(\vec{x}, \vec{y}) \geq \min_{\vec{y} \in \mathcal{Y}} F_0(\vec{x}, \vec{y}) \tag{2}$$

$$\geq \min_{\vec{x} \in \mathcal{X}} \min_{\vec{y} \in \mathcal{Y}} F_0(\vec{x}, \vec{y}) \tag{3}$$

where both lines follow from the definition of a minimum. The inequality above holds for every $\vec{x} \in \mathcal{X}$, so it holds for the value $\vec{x}$ that minimizes this quantity, i.e.,

$$\min_{\vec{x} \in \mathcal{X}} F_0(\vec{x}, \vec{y}) \geq \min_{\vec{x} \in \mathcal{X}} \min_{\vec{y} \in \mathcal{Y}} F_0(\vec{x}, \vec{y}). \tag{4}$$

This inequality also holds for every $\vec{y} \in \mathcal{Y}$, so

$$\min_{\vec{y} \in \mathcal{Y}} \min_{\vec{x} \in \mathcal{X}} F_0(\vec{x}, \vec{y}) \geq \min_{\vec{x} \in \mathcal{X}} \min_{\vec{y} \in \mathcal{Y}} F_0(\vec{x}, \vec{y}). \tag{5}$$

By symmetry, we can reverse our treatment of $\vec{x}$ and $\vec{y}$ and arrive at the reversed inequality

$$\min_{\vec{x} \in \mathcal{X}} \min_{\vec{y} \in \mathcal{Y}} F_0(\vec{x}, \vec{y}) \geq \min_{\vec{y} \in \mathcal{Y}} \min_{\vec{x} \in \mathcal{X}} F_0(\vec{x}, \vec{y}). \tag{6}$$

Since both (5) and (6) must hold, the expressions must be equal, as desired.

(b) Show that $p^\star \geq d^\star$, where

$$p^\star \doteq \min_{\vec{x} \in \mathcal{X}} \max_{\vec{y} \in \mathcal{Y}} F_0(\vec{x}, \vec{y}) \tag{7}$$

$$d^\star \doteq \max_{\vec{y} \in \mathcal{Y}} \min_{\vec{x} \in \mathcal{X}} F_0(\vec{x}, \vec{y}). \tag{8}$$

This statement is referred to as the *min-max theorem*.

Solution: By the definitions of minimization and maximization, we have that

$$L(\vec{y}) \doteq \min_{\vec{x}'} F_0(\vec{x}', \vec{y}) \leq F_0(\vec{x}, \vec{y}) \leq U(\vec{x}) \doteq \max_{\vec{y}'} F_0(\vec{x}, \vec{y}') \tag{9}$$

for every $\vec{x} \in \mathcal{X}$ and $\vec{y} \in \mathcal{Y}$, or more simply,

$$L(\vec{y}) \leq U(\vec{x}). \tag{10}$$

Since this inequality holds for all $\vec{x} \in \mathcal{X}$, it holds for the value of $\vec{x}$ that minimizes $U(\vec{x})$, and thus

$$p^\star = \min_{\vec{x} \in \mathcal{X}} U(\vec{x}) \geq L(\vec{y}). \tag{11}$$

Similarly, since the above holds for all $\vec{y} \in \mathcal{Y}$, it holds for the value of $\vec{y}$ that maximizes $L(\vec{y})$, and thus

$$p^\star \geq \max_{\vec{y} \in \mathcal{Y}} L(\vec{y}) = d^\star \tag{12}$$

as desired.