Quiz03

1. Diagonalization and Singular Value Decomposition

Let matrix $A = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

(a) Compute the eigenvalues and associated eigenvectors of A.

Solution: Eigenvalues can be computed by first calculating A's characteristic polynomial:

$$\begin{aligned}
\det(sI - A) &= \det\left(\begin{bmatrix} s & -1 \\ -\frac{1}{2} & s - \frac{1}{2} \end{bmatrix}\right) && (1)\\
&= s\left(s - \frac{1}{2}\right) - (-1)\left(-\frac{1}{2}\right) && (2)\\
&= s^2 - \frac{1}{2}s - \frac{1}{2} && (3)\\
&= \left(s - \frac{1}{4}\right)^2 - \frac{1}{16} - \frac{1}{2} && (4)\\
&= \left(s - \frac{1}{4}\right)^2 - \frac{9}{16} && (5)\\
&= \left(s - \frac{1}{4} - \frac{3}{4}\right)\left(s - \frac{1}{4} + \frac{3}{4}\right) \qquad a^2 - b^2 = (a-b)(a+b) && (6)\\
&= (s-1)\left(s + \frac{1}{2}\right). && (7)
\end{aligned}$$

The eigenvalues of A are thus $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2}$, the values of s at which $\det(sI - A) = 0$.

The eigenvectors associated with each eigenvalue λ can be calculated as values of $\vec{x} = \begin{bmatrix} x_a \\ x_b \end{bmatrix}$ for which $A\vec{x} = \lambda\vec{x}$, namely:

$$A\vec{x} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x_a \\ x_b \end{bmatrix} = \begin{bmatrix} x_b \\ \frac{x_a + x_b}{2} \end{bmatrix} \tag{8}$$

$$A\vec{x}_1 = \vec{x}_1 \iff x_b = x_a \iff \vec{x}_1 = \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \alpha_1 \neq 0 \in \mathbb{R}. \tag{9}$$

$$A\vec{x}_2 = -\frac{1}{2}\vec{x}_2 \iff x_b = -\frac{1}{2}x_a \iff \vec{x}_2 = \alpha_2 \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix},\ \alpha_2 \neq 0 \in \mathbb{R}. \tag{10}$$

Note that the expressions above are valid eigenvectors for any nonzero values of α_1 and α_2.

(b) Express A as $P\Lambda P^{-1}$, where Λ is a diagonal matrix and $PP^{-1} = I$. State P, Λ, and P^{-1} explicitly.

Solution: Combining the calculations in part 1(a), we have that

$$A\begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{x}_1 & \lambda_2\vec{x}_2 \end{bmatrix} = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{11}$$

For our calculations, we will use the eigenvalues and eigenvectors from part 1(a) with $\alpha_1 = \alpha_2 = 1$. (Your calculations may differ here; any nonzero values for α_1 and α_2 are permissible, and will result in scaled values of P and P^{-1}.) Filling in eigenvalue and eigenvector values, we have:

$$A\begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, \tag{12}$$

and rearranging,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}^{-1}. \tag{13}$$

Calculating the latter inverse explicitly, we have

$$\begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}^{-1} = -\frac{2}{3} \begin{bmatrix} -\frac{1}{2} & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \text{ because } \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{14}$$

so finally,

$$A = P\Lambda P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}. \tag{15}$$

This is known as the *eigenvalue decomposition*, or *eigendecomposition*, of matrix A; for a more extensive description of this decomposition, see Calafiore & El Ghaoui section 3.5.

(c) Compute $\lim_{k\to\infty} A^k$.

Solution: Using the diagonalization of A from part 1(b), we have:

$$\begin{aligned}
A &= P\Lambda P^{-1} && (16)\\
A^k &= (P\Lambda P^{-1})^k && (17)\\
&= (P\Lambda P^{-1})(P\Lambda P^{-1})\dots(P\Lambda P^{-1}) \qquad (k \text{ times}) && (18)\\
&= P\Lambda \underbrace{P^{-1}P}_{I} \Lambda P^{-1} \dots P\Lambda P^{-1} && (19)\\
&= P\Lambda^k P^{-1} && (20)\\
&= \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}^k \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix} && (21)\\
&= \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (-\frac{1}{2})^k \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}. && (22)
\end{aligned}$$

Finally, because $\lim_{k\to\infty} \left(-\frac{1}{2}\right)^k = 0$, we have

$$\lim_{k\to\infty} A^k = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}. \tag{23}$$

(d) Give the singular values σ_1 and σ_2 of A.

Solution: Each singular value σ_i of A can be calculated as $\sigma_i = \sqrt{\lambda_i(AA^\top)} = \sqrt{\lambda_i(A^\top A)}$. (This is because A's singular value decomposition, canonically written $A = U\Sigma V^\top$, can be multiplied by a transposed version to give $AA^\top = U\Sigma^2 U^\top$, where Σ^2 is a diagonal matrix containing the squared singular values of A and $UU^\top = I$. For a thorough treatment of SVD, see Calafiore & El Ghaoui chapter 5.)

To find A's singular values, we thus perform the same calculation used in part 1(a) to find each $\lambda_i(AA^\top) = \sigma_i^2$:

$$\begin{aligned}
AA^\top &= \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} && (24)\\
&= \frac{1}{2} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}. && (25)
\end{aligned}$$

$$\det\left((sI - AA^\top)\right) = \det\left(\begin{bmatrix} s-1 & -\frac{1}{2} \\ -\frac{1}{2} & s-\frac{1}{2} \end{bmatrix}\right) \tag{26}$$

$$= (s-1)\left(s-\frac{1}{2}\right) - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) \tag{27}$$

$$= s^2 - s - \frac{1}{2}s + \frac{1}{2} - \frac{1}{4} \tag{28}$$

$$= s^2 - \frac{3}{2}s + \frac{1}{4} \tag{29}$$

$$= \left(s - \frac{3}{4}\right)^2 - \frac{9}{16} + \frac{1}{4} \tag{30}$$

$$= \left(s - \frac{3}{4}\right)^2 - \frac{5}{16} \tag{31}$$

$$= \left(s - \frac{3}{4} - \frac{\sqrt{5}}{4}\right)\left(s - \frac{3}{4} + \frac{\sqrt{5}}{4}\right) \qquad a^2 - b^2 = (a-b)(a+b) \tag{32}$$

$$= \left(s - \frac{3+\sqrt{5}}{4}\right)\left(s - \frac{3-\sqrt{5}}{4}\right) \tag{33}$$

$$= (s - \sigma_1^2)(s - \sigma_2^2). \tag{34}$$

Thus, the singular values of A are $\sigma_1 = \frac{\sqrt{3+\sqrt{5}}}{2}$ and $\sigma_2 = \frac{\sqrt{3-\sqrt{5}}}{2}$.