Quiz03

1. Strong Duality but No KKT

In this question, we will see an example of a problem where strong duality holds but the KKT conditions don't hold. Consider the following problem:

$$p^* = \min_{x_1, x_2 \in \mathbb{R}} \quad x_1^2 + x_2^2 \tag{1}$$
$$\text{s.t.} \quad (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1 \tag{2}$$
$$(x_1 - 1)^2 + (x_2 + 1)^2 \leq 1. \tag{3}$$

(a) Write the KKT conditions and solve them.

Solution: The KKT conditions are,

$$2x_1 + 2\lambda_1(x_1 - 1) + 2\lambda_2(x_1 - 1) = 0 \tag{4}$$
$$2x_2 + 2\lambda_1(x_2 - 1) + 2\lambda_2(x_2 + 1) = 0 \tag{5}$$
$$(x_1 - 1)^2 + (x_2 - 1)^2 \leq 1 \tag{6}$$
$$(x_1 - 1)^2 + (x_2 + 1)^2 \leq 1 \tag{7}$$
$$\lambda_1 \geq 0 \tag{8}$$
$$\lambda_2 \geq 0 \tag{9}$$
$$\lambda_1((x_1 - 1)^2 + (x_2 - 1)^2 - 1) = 0 \tag{10}$$
$$\lambda_2((x_1 - 1)^2 + (x_2 + 1)^2 - 1) = 0. \tag{11}$$

From Equations 6 and 7, we have $x_1 = 1, x_2 = 0$. Substituting in 4 we get,

$$2 = 0, \tag{12}$$

which is clearly impossible so the KKT conditions have no solution.

(b) Write the Lagrangian $L(x_1, x_2, \lambda_1, \lambda_2)$ where λ_1, λ_2 are the Lagrangian multipliers of the constraints, and solve for x_1^* and x_2^* that minimize the Lagrangian:

$$x_1^*, x_2^* \in \operatorname*{argmin}_{x_1, x_2 \in \mathbb{R}} L(x_1, x_2, \lambda_1, \lambda_2) \tag{13}$$

Note that x_1^* and x_2^* should be in terms of the dual variables λ_1 and λ_2.

Solution: The Lagrangian is given by,

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = x_1^2 + x_2^2 + \lambda_1((x_1 - 1)^2 + (x_2 - 1)^2 - 1) + \lambda_2((x_1 - 1)^2 + (x_2 + 1)^2 - 1) \tag{14}$$
$$= (1 + \lambda_1 + \lambda_2)x_1^2 + (1 + \lambda_1 + \lambda_2)x_2^2 - 2(\lambda_1 + \lambda_2)x_1 - 2(\lambda_1 - \lambda_2)x_2 + \lambda_1 + \lambda_2. \tag{15}$$

Setting the derivative with respect to x_1, x_2 to 0 we get that $\mathcal{L}$ achieves minimum at,

$$x_1 = \frac{\lambda_1 + \lambda_2}{1 + \lambda_1 + \lambda_2}, \tag{16}$$
$$x_2 = \frac{\lambda_1 - \lambda_2}{1 + \lambda_1 + \lambda_2}. \tag{17}$$