

1. Strong Duality but No KKT

In this question, we will see an example of a problem where strong duality holds but the KKT conditions don't hold. Consider the following problem:

$$p^* = \min_{x_1, x_2 \in \mathbb{R}} x_1^2 + x_2^2 \quad (1)$$

$$\text{s.t. } (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1 \quad (2)$$

$$(x_1 - 1)^2 + (x_2 + 1)^2 \leq 1. \quad (3)$$

(a) Write the KKT conditions and solve them.

Solution: The KKT conditions are,

$$2x_1 + 2\lambda_1(x_1 - 1) + 2\lambda_2(x_1 - 1) = 0 \quad (4)$$

$$2x_2 + 2\lambda_1(x_2 - 1) + 2\lambda_2(x_2 + 1) = 0 \quad (5)$$

$$(x_1 - 1)^2 + (x_2 - 1)^2 \leq 1 \quad (6)$$

$$(x_1 - 1)^2 + (x_2 + 1)^2 \leq 1 \quad (7)$$

$$\lambda_1 \geq 0 \quad (8)$$

$$\lambda_2 \geq 0 \quad (9)$$

$$\lambda_1((x_1 - 1)^2 + (x_2 - 1)^2 - 1) = 0 \quad (10)$$

$$\lambda_2((x_1 - 1)^2 + (x_2 + 1)^2 - 1) = 0. \quad (11)$$

From Equations 6 and 7, we have $x_1 = 1, x_2 = 0$. Substituting in 4 we get,

$$2 = 0, \quad (12)$$

which is clearly impossible so the KKT conditions have no solution.

(b) Write the Lagrangian $L(x_1, x_2, \lambda_1, \lambda_2)$ where λ_1, λ_2 are the Lagrangian multipliers of the constraints, and solve for x_1^* and x_2^* that minimize the Lagrangian:

$$x_1^*, x_2^* \in \operatorname{argmin}_{x_1, x_2 \in \mathbb{R}} L(x_1, x_2, \lambda_1, \lambda_2) \quad (13)$$

Note that x_1^* and x_2^* should be in terms of the dual variables λ_1 and λ_2 .

Solution: The Lagrangian is given by,

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = x_1^2 + x_2^2 + \lambda_1((x_1 - 1)^2 + (x_2 - 1)^2 - 1) + \lambda_2((x_1 - 1)^2 + (x_2 + 1)^2 - 1) \quad (14)$$

$$= (1 + \lambda_1 + \lambda_2)x_1^2 + (1 + \lambda_1 + \lambda_2)x_2^2 - 2(\lambda_1 + \lambda_2)x_1 - 2(\lambda_1 - \lambda_2)x_2 + \lambda_1 + \lambda_2. \quad (15)$$

Setting the derivative with respect to x_1, x_2 to 0 we get that \mathcal{L} achieves minimum at,

$$x_1 = \frac{\lambda_1 + \lambda_2}{1 + \lambda_1 + \lambda_2}, \quad (16)$$

$$x_2 = \frac{\lambda_1 - \lambda_2}{1 + \lambda_1 + \lambda_2}. \quad (17)$$