Quiz03

1. Dual of a linear program

Consider a standard linear program P:

$$\begin{aligned} \min_{\vec{x}} \quad & \vec{c}^\top \vec{x} && (1)\\ \text{s.t.} \quad & A\vec{x} = \vec{b} && (2)\\ & \vec{x} \geq \vec{0}. && (3) \end{aligned}$$

where $\vec{x}, \vec{c} \in \mathbb{R}^n, \vec{b} \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$. $\vec{x} \geq \vec{0}$ means $x_i \geq 0$ for all $i = 1, \ldots, n$.

(a) Formulate the Lagrangian of the problem P, and write the dual problem. You do not need to solve the dual problem.

Note: The dual problem should not have the variable $\vec{x}$.

Solution: Denote the dual variable associated with the equality constraint in P by $\vec{\nu} \in \mathbb{R}^m$, and the dual variable associated with the inequality by $\vec{\lambda} \in \mathbb{R}^n$. The Lagrangian of P is given by

$$\mathcal{L}(\vec{x}, \vec{\nu}, \vec{\lambda}) = \vec{c}^\top \vec{x} + \vec{\nu}^\top (A\vec{x} - \vec{b}) + \lambda^\top (-\vec{x}) = -\vec{b}^\top \vec{\nu} + (\vec{c} + A^\top \vec{\nu} - \vec{\lambda})^\top \vec{x}. \tag{4}$$

To get the dual problem, we take the minimum of the Lagrangian with respect to our primal variable $\vec{x}$, i.e.,

$$g(\vec{\nu}) = \min_{\vec{x}} \mathcal{L}(\vec{x}, \vec{\nu}, \vec{\lambda}) = \begin{cases} -\vec{b}^\top \vec{\nu} & \text{if } \vec{c} + A^\top \vec{\nu} - \vec{\lambda} = \vec{0} \\ -\infty & \text{otherwise.} \end{cases} \tag{5}$$

The dual problem is then given by D:

$$\begin{aligned} \max_{\vec{\nu}, \vec{\lambda}} \quad & -\vec{b}^\top \vec{\nu} && (6)\\ \text{s.t.} \quad & \vec{c} + A^\top \vec{\nu} - \vec{\lambda} = \vec{0} && (7)\\ & \vec{\lambda} \geq \vec{0}. && (8) \end{aligned}$$

We can simplify this further by noting that $\vec{\lambda}$ is constrained to be $\vec{\lambda} \geq \vec{0}$, since it is the dual variable associated with an inequality constraint. Then, $\vec{c} + A^\top \vec{\nu} - \vec{\lambda} = \vec{0} \iff \vec{c} + A^\top \vec{\nu} = \vec{\lambda} \geq \vec{0}$, and we can eliminate $\vec{\lambda}$ to get

$$\begin{aligned} \max_{\vec{\nu}} \quad & -\vec{b}^\top \vec{\nu} && (9)\\ \text{s.t.} \quad & \vec{c} + A^\top \vec{\nu} \geq \vec{0}. && (10) \end{aligned}$$