Quiz03

1. Lagrangian Dual of a QP

Consider the standard form of a convex quadratic program, with $Q \succ 0$:

$$\min_{\vec{x}} \quad \frac{1}{2}\vec{x}^\top Q\vec{x} \tag{1}$$
$$\text{s.t.} \quad A\vec{x} \leq \vec{b} \tag{2}$$

(a) Write the Lagrangian function $\mathcal{L}(\vec{x}, \vec{\lambda})$.

Solution:

$$\mathcal{L}(\vec{x}, \vec{\lambda}) = \frac{1}{2}\vec{x}^\top Q\vec{x} + \vec{\lambda}^\top (A\vec{x} - \vec{b}) \tag{3}$$

(b) Write the Lagrangian dual function, $g(\vec{\lambda})$.

Solution:

$$g(\vec{\lambda}) = \inf_{\vec{x}} \ \mathcal{L}(\vec{x}, \vec{\lambda}) \tag{4}$$

We can find this infimum by setting $\nabla_{\vec{x}}\mathcal{L}(\vec{x}^\star, \vec{\lambda}) = 0$:

$$Q\vec{x}^\star + A^\top\vec{\lambda} = 0 \implies \vec{x}^\star = -Q^{-1}A^\top\vec{\lambda} \tag{5}$$

Substituting, we get

$$\begin{aligned} g(\vec{\lambda}) &= \mathcal{L}(\vec{x}^\star, \vec{\lambda}) && (6)\\ &= \frac{1}{2}\vec{\lambda}^\top AQ^{-\top}A^\top\vec{\lambda} - \vec{\lambda}^\top AQ^{-1}A^\top\vec{\lambda} - \vec{\lambda}^\top\vec{b} && (7)\\ &= -\frac{1}{2}\vec{\lambda}^\top AQ^{-1}A^\top\vec{\lambda} - \vec{\lambda}^\top\vec{b} && (8) \end{aligned}$$

(c) Show that the Lagrangian dual problem is convex by writing it in "standard QP form" – that is, in a form similar to the original problem. Is the Lagrangian dual problem convex in general?

Solution: The Lagrangian dual problem writes

$$\max_{\vec{\lambda}\geq 0} \ g(\vec{\lambda}) = \max_{\vec{\lambda}\geq 0} \ -\frac{1}{2}\vec{\lambda}^\top AQ^{-1}A^\top\vec{\lambda} - \vec{\lambda}^\top\vec{b}, \tag{9}$$

the maximization of a concave function of $\vec{\lambda}$ over the convex region given by the non-negative orthant $\vec{\lambda} \geq 0$. The dual problem is therefore convex.

While in this problem, the primal problem was convex, it turns out that the Lagrangian dual problem is a convex problem even when the primal is not. To see this, examine its general form:

$$\max_{\vec{\lambda}\geq 0}\min_{\vec{x}} \mathcal{L}(\vec{x}, \vec{\lambda}) = \max_{\vec{\lambda}\geq 0}\min_{\vec{x}} \left[f_0(\vec{x}) + \sum_{i=1}^{n} \lambda_i f_i(\vec{x}) \right] \tag{10}$$

This represents the pointwise minimum of affine functions of $\vec{\lambda}$, which we know to be concave. The resulting maximization problem of a concave objective in $\vec{\lambda}$ over the convex region $\vec{\lambda} \geq 0$ is then a convex optimization problem!